Part III Model Theory, Lecture 16, 16 Nov

Theorem 16.1 (Vaught 1959)

Suppose T is a complete $\mathcal{L}$-theory, $\mathcal{L}$ ctble. $I(T, \aleph_0) \neq 2$.

Proof Assume $I(T, \aleph_0) = 2$. By Prop 14.4, T has a prime model $\mathcal{M}$ and a ctble saturated model $\mathcal{N}$. By Thm 15.2, $\exists$ some non-isolated $p \in S_n(T)$ for some $n \geq 1$. So $\mathcal{M}$ omits p, and $\exists\ \bar{a} \in N^n$ realizing p. Let $T^* = Th_{\bar{a}}(\mathcal{N})$. Then $\mathcal{N}$ is still saturated as an $\mathcal{L}_{\bar{a}}$-structure (ES3 #3). So T^* has a prime model $\mathcal{B}$ by Thm 14.2(b). Let $\mathcal{A} = \mathcal{B}\restriction\mathcal{L} \models T$. So $\mathcal{A}$ realizes p. So $\mathcal{A} \not\cong \mathcal{M}$. So $\mathcal{A} \cong \mathcal{N}$. So $\mathcal{B} = \mathcal{N}$ (ES3 #3). So the prime model of T^* is saturated. So T^* is $\aleph_0$-categorical by Thm 15.2. So T is $\aleph_0$-categorical by Cor 15.4. ↯. $\square$

Uncountable Saturated Models

Theorem 16.2 For any infinite $\mathcal{M}$ and $\kappa \geq |\mathcal{L}| + \aleph_0$, there is an $\mathcal{N} \succeq \mathcal{M}$ st $\mathcal{N}$ is κ^+-saturated and $|N| \leq |M|^\kappa$.

Proof Fix $\kappa \geq |\mathcal{L}| + \aleph_0$. Notation: $X \subseteq_\kappa Y$ means $X \subseteq Y$ and $|X| \leq \kappa$.

Claim: For any $\mathcal{M}$, there is $\mathcal{N} \succeq \mathcal{M}$ st $|N| \leq |M|^\kappa$ and $\mathcal{N}$ realizes all types in $S_1^{\mathcal{M}}(A)$ $\forall$ $A \subseteq_\kappa M$.

Proof: # subsets of M of size $\leq \kappa$ is $|M|^\kappa$ and if $|A| \leq \kappa$, then $|S_1^{\mathcal{M}}(A)| \leq 2^\kappa \leq |M|^\kappa$. Enumerate all such types as $(p_\alpha)_{\alpha < |M|^\kappa}$ (α ordinal). Build elementary chain $(\mathcal{M}_\alpha)_{\alpha < |M|^\kappa}$ st $\mathcal{M}_0 = \mathcal{M}$, for limit α, $\mathcal{M}_\alpha = \bigcup_{i<\alpha} \mathcal{M}_i$, and $\mathcal{M}_{\alpha+1} \succeq \mathcal{M}_\alpha$ realizes p_α st $|M_{\alpha+1}| \leq |M_\alpha| + |\mathcal{L}|$ (by Prop 8.4). Let $\mathcal{N} = \bigcup_\alpha \mathcal{M}_\alpha$. Then $|N| \leq |M|^\kappa$ and $\mathcal{N}$ realizes all p_α. //

Fix $\mathcal{M}$. Now build elementary chain $(\mathcal{N}_\alpha)_{\alpha < \kappa^+}$ st $|N_\alpha| \leq |M|^\kappa$ and

1. $\mathcal{N}_0 = \mathcal{M}$

2. α limit, $\mathcal{N}_\alpha = \bigcup_{i<\alpha} \mathcal{N}_i$.

3. Given $\alpha < \kappa^+$, let $\mathcal{N}_{\alpha+1} \succeq \mathcal{N}_\alpha$ st $|N_{\alpha+1}| \leq |N_\alpha|^\kappa$ and $\mathcal{N}_{\alpha+1}$ realizes all types over all set $A \subseteq_\kappa \mathcal{N}_\alpha$. (by the Claim).

Let $\mathcal{N} = \bigcup_{\alpha<\kappa^+} \mathcal{N}_\alpha$. By induction on α, $|N| \leq |M|^\kappa$. $\mathcal{N}$ is κ^+-saturated since $A \subseteq_\kappa N \Rightarrow A \subseteq_\kappa N_\alpha$ for some $\alpha < \kappa^+$. $\square$

Let T be an $\mathcal{L}$-theory with infinite models.

Corollary 16.3: If $\kappa \geq |\mathcal{L}|^{+\aleph_0}$ and $2^\kappa = \kappa^+$, then T has a saturated model of size κ.

Pf: By 16.2, T has a κ^+-saturated model of size $(|\mathcal{L}| + \aleph_0)^\kappa = 2^\kappa = \kappa^+$.

Fact: If $\kappa > |\mathcal{L}| + \aleph_0$ is regular and $2^\lambda \leq \kappa$ $\forall \lambda < \kappa$, then T has a saturated model of size κ.

Basic Facts

1) If $\mathcal{M} \equiv \mathcal{N}$, $|M| = |N|$, and $\mathcal{M}, \mathcal{N}$ are saturated, then $\mathcal{M} \cong \mathcal{N}$. (ES3 #4).

2) Suppose $\kappa > |\mathcal{L}| + \aleph_0$. Then $\mathcal{M}$ is κ-saturated iff $\mathcal{M}$ is κ-homogeneous and
$$\underbrace{\forall\ \mathcal{N} \equiv \mathcal{M},\ \text{if } |N| < \kappa \text{ then } \mathcal{N} \text{ elementarily embeds in } \mathcal{M}}_{\text{"}\mathcal{M}\text{ is }\kappa\text{-universal"}}$$

Stability Let T be a complete theory with infinite models.

Def 16.4 Given $\kappa \geq |\mathcal{L}| + \aleph_0$, we say T is κ-stable if $\forall\ \mathcal{M} \models T$, $|M| = \kappa$ we have $|S_1(M)| = \kappa$. T is stable if it is κ-stable for some $\kappa \geq |\mathcal{L}| + \aleph_0$.

Example 16.5

① ACF_p, TFDAG are κ-stable $\forall\ \kappa \geq \aleph_0$. (see Example 9.3)

② $T = Th(\mathbb{Z}, +, 0, 1, (\equiv_n)_{n\geq 2})$ where $x \equiv_n y$ iff $\exists z\,(x-y = nz)$. T has QE.

Fix $M \models T$. Given $f : \{\text{primes}\} \to \mathbb{N}$ st $0 \leq f(n) < n$, we have a complete 1-type

$$p_f = \{x \neq a : a \in M\} \cup \{x \equiv_n f(n) : n \text{ is prime}\} \in S_1(M).$$

By QE, $S_1(M) = \{tp(a/M) : a \in M\} \cup \{p_f\}_f$. So $|S_1(M)| = |M| + 2^{\aleph_0}$

Thus T is κ-stable iff $\kappa \geq 2^{\aleph_0}$.

③ If $M \models RG$ then $|S_1(M)| = 2^{|M|}$ (Ex 9.4). So RG is not stable.