Part III Model Theory, Lecture 17, 18 Nov

T complete, infinite models.

Recall: A cardinal κ is regular if every unbounded subset of κ has size κ.

E.g. $\aleph_0$, $\aleph_1$, $2^{\aleph_0}$ are regular. $\aleph_\omega$ not regular, e.g. $\{\aleph_n : n \geq 1\}$ is unbounded.

Theorem 17.1 Suppose T is κ-stable and κ is regular. Then T has a saturated model of size κ.

Proof

Step 1: If $M \models T$, $|M| = \kappa$, then $\exists N \succeq M$ st $|N| = \kappa$, and N realizes all 1-types over M.

Pf: Enumerate $S_1(M) = \{p_\alpha : \alpha < \kappa\}$ by κ-stability. Build chain.

Step 2: Build $(M_\alpha)_{\alpha<\kappa}$ st $|M_0| = \kappa$. $M_{\alpha+1} \succeq M_\alpha$ realizes all 1-types over M_α, and $|M_{\alpha+1}| = \kappa$. Let $M = \bigcup_{\alpha<\kappa} M_\alpha$. Then $|M| = \kappa$. If $A \subset M$, $|A| < \kappa$, then $A \subseteq M_\alpha$ for some $\alpha < \kappa$ since κ is regular. So M realizes all 1-types over A. □

Theorem 17.2: L ctble. Suppose T is $\aleph_0$-stable. Then T is κ-stable $\forall\, \kappa \geq \aleph_0$.

Proof

Fix $\kappa \geq \aleph_0$. Assume we have $M \models T$, $|M| = \kappa$, st $|S_1(M)| > \kappa$. Then $\exists$ L_M-formula $\varphi(x)$ st $|[\varphi(x)]| > \kappa$. $\#L_M$-formulas $\leq \kappa$ $\quad [x=x] = S_1(M)$

Claim: For an L_M-formula $\varphi(x)$ if $|[\varphi(x)]| > \kappa$ then $\exists$ L_M-formula $\psi(x)$ st $|[\varphi \wedge \psi]| > \kappa$, $|[\varphi \wedge \neg\psi]| > \kappa$.

Pf: Adapt the claim from the proof of Lemma 14.1.

Now build $\{\varphi_\sigma\}_{\sigma \in 2^{<\omega}}$ st $[\varphi_\sigma]$ is partitioned $[\varphi_{\sigma 0}] \cup [\varphi_{\sigma 1}]$ and $[\varphi_\sigma] \neq \emptyset$.

Let $N \preceq M$ st $|N| = \aleph_0$ and N contains all parameters used in all φ_σ's.

For $\alpha \in 2^\omega$, find $p_\alpha \in S_1(N)$ st $\varphi_{\alpha \restriction i} \in p_\alpha$ $\forall\, i \geq 0$. $|S_1(N)| = 2^{\aleph_0}$. T is not $\aleph_0$-stable. □

Corollary 17.3 If T is $\aleph_0$-stable then then T has a saturated model of size κ $\forall$ regular $\kappa \geq \aleph_0$.

Fact: $\aleph_0$-stable theories have saturated models of all infinite cardinalities.

Def 17.4 Fix $\mathcal{M} \models T$, and an $\mathcal{L}$-formula $\varphi(x_1, \dots, x_m, y_1, \dots, y_n)$. Then $p \in S_m(M)$ is <u>definable wrt $\varphi(\bar{x}, \bar{y})$</u> if $\exists$ $\mathcal{L}_M$-formula $\psi(y_1, \dots, y_n)$ st $\forall$ $\bar{b} \in M^n$,

$$\varphi(\bar{x}, \bar{b}) \in p \quad \text{iff} \quad \mathcal{M} \models \psi(\bar{b}).$$

We say $p \in S_m(M)$ is <u>definable</u> if it is definable wrt any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ (any $\bar{y}$).

Example 17.5

1) $p = tp(a/M)$ where $a \in \mathcal{M}$. Given $\varphi(x, \bar{y})$ let $\psi(\bar{y})$ be $\varphi(a, \bar{y}) \leftarrow \mathcal{L}_M$-formula.

2) T is DLO. $\mathcal{M}$ is $\mathbb{Q}$. Choose $p \in S_1(\mathbb{Q})$ st $x < b$ is in p iff $\sqrt{2} < b$.
   Let $\varphi(x, y)$ be $x < y$. Then $\{b \in \mathbb{Q} : \varphi(x, b) \in p\} = (-\infty, \sqrt{2}) \cap \mathbb{Q}$.
   By QE any definable subset of $\mathbb{Q}$ is a Boolean comb. of intervals with endpoints in $\mathbb{Q}$.

Notation: Let x be a tuple of variables. M^x denotes $M^{|x|}$.

Def 17.6: Let $\varphi(x, y)$ be an $\mathcal{L}$-formula [x, y tuples]. Then $\varphi(x, y)$ has the <u>order property</u> wrt T if $\exists$ $\mathcal{M} \models T$ and $(a_i)_{i \geq 0}$, $(b_i)_{i \geq 0}$ st $a_i \in M^x$, $b_i \in M^y$, $\forall i \geq 0$

and $\forall i, j$, $\quad \mathcal{M} \models \varphi(a_i, b_j)$ iff $i \leq j$.

E.g. In $\mathbb{Q}$, $x \leq y$ has the OP (wrt DLO) Let $a_i = b_i = i$.

<u>Fundamental Theorem of Stability</u> (Shelah 1976) TFAE

1) T is stable.

2) For any $\mathcal{M} \models T$, any $p \in S_n(M)$ is definable.

3) No $\mathcal{L}$-formula has the OP wrt T.

FTS2 $\Rightarrow$ FTS1 Assume 2. Fix $\kappa \geq |\mathcal{L}| + \aleph_0$ st $\kappa^{|\mathcal{L}|+\aleph_0} = \kappa$ (e.g. $\kappa = 2^{|\mathcal{L}|+\aleph_0}$)

We show T is κ-stable. Fix $M \models T$, $|M| = \kappa$. Let $X = \{\mathcal{L}\text{-formulas } \varphi(x, \bar{y})\}$ and $Y = \{\text{all } \mathcal{L}_M\text{-formulas } \psi(\bar{y})\}$ (any $\bar{y}$). Given $p \in S_1(M)$, define $F_p : X \to Y$ st $F_p(\varphi(x, \bar{y}))$ witnesses that p is definable wrt $\varphi(x, \bar{y})$. Then $p \mapsto F_p$ is an injective function from $S_1(M)$ to Y^X. So $|S_1(M)| \leq |Y^X| = \kappa^{|\mathcal{L}|+\aleph_0} = \kappa$. $\square$