Part III Model Theory, Lecture 18, 20 Nov

T complete, infinite models.

FTS1 $\Rightarrow$ FTS3 Suppose $\varphi(x,y)$ has OP wrt T. (x,y tuples). Fix $\kappa \geq |\mathcal{L}| + \aleph_0$.

WTS T is not κ-stable.

Claim: There is a LO $(I,<)$ st $|I| > \kappa$ and there is $J \subseteq I$ st $|J| = \kappa$ and J is dense.

Proof: Fix minimal $\lambda \in \kappa$ st $2^\lambda > \kappa$. Let $I = \mathbb{Q}^\lambda$. Given distinct $f, g \in I$ set $f<g$ iff $f(\alpha) < g(\alpha)$ where $\alpha < \lambda$ is least st $f(\alpha) \neq g(\alpha)$.
Let $J = \{f \in I : \exists\, \alpha < \lambda$ st $f(x) = 0\ \forall\, x \geq \alpha\}$. $|J| \leq \max_{\alpha<\lambda} 2^\alpha \leq \kappa$. //

Let I be as in the claim. Consider

$$T \cup \{\varphi(x_i, y_j) : i,j \in I : i \leq j\} \cup \{\neg\varphi(x_i, y_j) : i,j \in I,\ i > j\}.$$

This is finitely satisfiable since $\varphi(x,y)$ has OP wrt T. So $\exists$ $\mathcal{N} \models T$ and $(a_i)_{i\in I}$ and $(b_i)_{i \in I}$, $a_i \in N^x$, $b_i \in N^y$ and $\mathcal{N} \models \varphi(a_i, b_j)$ iff $i \leq j$. By DLST $\exists$ $\mathcal{M} \preceq \mathcal{N}$ st $|M| = \kappa$ and $b_i \in M^y$ $\forall$ $i \in J$. We show $|S_x(M)| \geq |I| > \kappa$, and thus T is not κ-stable (ES3 #5). For $i \in I$, let $p_i = tp(a_i/M)$.
Fix $i,j \in I$ st $i<j$. $\exists$ $k \in J$ st $i<k<j$. So $\varphi(x, b_k) \in p_i$ and $\varphi(x, b_k) \notin p_j$. So $p_i \neq p_j$. $\square$

FTS3 $\Rightarrow$ FTS2 Fix an $\mathcal{L}$-formula $\varphi(x,y)$ st φ does not have OP wrt T. Fix $\mathcal{M} \models T$.
We show that any $p \in S_x(M)$ is definable wrt $\varphi(x,y)$.
Let $X = S_x(M)$ and $Y = S_y(M)$. Let $A = \{tp(a/M) : a \in M^x\} \subseteq X$
A is dense in X: Given an $\mathcal{L}_M$-formula $\psi(x)$, if $[\psi(x)] \neq \emptyset$ then $\mathcal{M} \models \exists x \psi(x)$, so $\exists$ $a \in M^x$ st $\mathcal{M} \models \psi(a)$, so $tp(a/M) \in [\psi(x)] \cap A$.
Let $B = \{tp(b/M) : b \in M^y\}$. Then B is dense Y.

Identify A with M^x and B with M^y. Let $2 = \{0,1\}$ (discrete space).

Define $f : A \times B \to 2$ st $f(a,b) = 1$ iff $M \models \varphi(a,b)$.

<u>Notation:</u> Given $a \in A$, $b \in B$, let $f_b : A \to 2$ and $f^a : B \to 2$ be the corresponding fiber functions (e.g. $f_b(a) = f(a,b)$).

Given $b \in B$, let $\hat{f}_b : X \to 2$ st for $p \in X = S_x(M)$, $\hat{f}_b(p) = 1$ iff $\varphi(x,b) \in p$.

$\hat{f}_b$ extends f_b: Given $a \in A$, $\hat{f}_b(a) = 1$ iff $\varphi(x,b) \in tp(a/M)$ iff $M \models \varphi(a,b)$
iff $f_b(a) = 1$.

$\hat{f}_b$ is continuous: $(\hat{f}_b)^{-1}(\{1\}) = [\varphi(x,b)]$ $(\hat{f}_b)^{-1}(\{0\}) = [\neg\varphi(x,b)]$.

Similarly given $a \in A$, $\hat{f}^a : Y \to 2$ st $\hat{f}^a(q) = 1$ iff $\varphi(a,y) \in q$.

<u>Main Claim</u>: There is a separately continuous function $F : X \times Y \to 2$ extending f.

Suppose this holds. Fix $p \in S_x(M) = X$. WTS p is definable wrt $\varphi(x,y)$.

$F^p : Y \to 2$ is continuous. Set $D = (F^p)^{-1}(\{1\})$. D is clopen in $Y = S_y(M)$.

So $D = [\psi(y)]$ for some $\mathcal{L}_M$-formula $\psi(y)$. (ES2 #7). Fix $b \in M^y = B$

$\varphi(x,b) \in p$ iff $\hat{f}_b(p) = 1$

iff $F_b(p) = 1$ $\left[\begin{array}{l} F_b,\ \hat{f}_b \text{ are continuous extensions } f_b \text{ and} \\ A \text{ is dense.}\end{array}\right]$

iff $F^p(b) = 1$

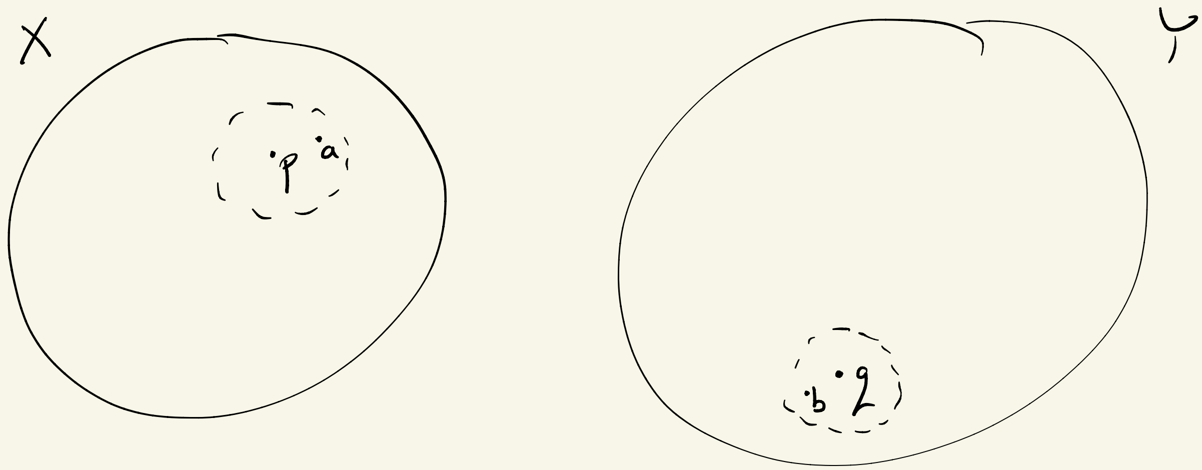
iff $tp(b/M) \in D = [\psi(y)]$

iff $M \models \psi(b)$.

<u>Proof of the Main Claim</u>

<u>Goal</u> ⊛ $\forall\, p \in X,\ q \in Y$ $\exists$ open nbhds U_{pq} of p & V_{pq} of q st

$\forall\ a \in A \cap U_{pq}$ and $b \in B \cap V_{pq}$, $\quad \hat{f}_b(p) = \hat{f}^a(q) =: t_{pq}$.



Given $\circledast$: define $F : X \times Y \to \{0,1\}$ st $F(p,q) = t_{pq}$.

F extends f: For $a \in A$, $b \in B$, $\quad a \in A \cap U_{ab} \Rightarrow t_{ab} = \hat{f}^a(b) = f(a,b)$.

F is separately continuous: e.g. $p \in X$, $t \in 2$

$$\left(F^p\right)^{-1}(\{t\}) = \bigcup \{ V_{pq} : q \in Y,\ t_{pq} = t \}.$$