Part III Model Theory, Lecture 19, 23 Nov

Setting T, $\varphi(x,y)$ no OP wrt T, $M \models T$

$X = S_x(M)$, $Y = S_y(M)$, $A = M^x \hookrightarrow X$, $B = M^y \hookrightarrow Y$

$f : A \times B \to 2$ st $f(a,b) = 1$ iff $M \models \varphi(a,b)$

$\hat{f}_b : X \to 2$ st $\hat{f}_b(p) = 1$ iff $\varphi(x,b) \in p$.

$\hat{f}^a : Y \to 2$ st $\hat{f}^a(q) = 1$ iff $\varphi(a,y) \in q$.

Goal (*): $\forall p \in X, q \in Y$ $\exists$ open nbhds U_{pq} of p and V_{pq} of q st

$\forall a \in A \cap U_{pq}$, $b \in B \cap V_{pq}$, $\hat{f}_b(p) = \hat{f}^a(q)$.

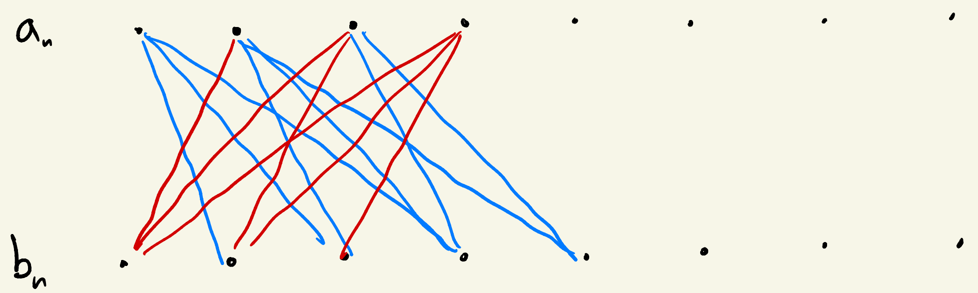
Suppose (*) fails. Fix $p \in X$, $q \in Y$ st $\forall$ open nbhds $U \ni p$, $V \ni q$, $\exists a \in A \cap U$, $b \in B \cap V$ st $\hat{f}_b(p) \neq \hat{f}^a(q)$.

We build $(a_n)_{n \geq 0}$ from $A = M^x$ and $(b_n)_{n \geq 0}$ from $B = M^y$ st $\forall n$

1. $\hat{f}_{b_n}(p) \neq \hat{f}^{a_n}(q)$
2. $\forall i < n$, $f(a_n, b_i) = \hat{f}_{b_i}(p)$
3. $\forall i < n$, $f(a_i, b_n) = \hat{f}^{a_i}(q)$.

Suppose we have this. Pass to subsequences so that $(\hat{f}_{b_n}(p))_n$ and $(\hat{f}^{a_n}(q))_n$ are constant.

Case 1: $\forall n$ $\hat{f}_{b_n}(p) = 0$ and $\hat{f}^{a_n}(q) = 1$



So $\mathcal{M} \models \varphi(a_i, b_j)$ for all $i<j$ and $\mathcal{M} \models \neg\varphi(a_i, b_j)$ for all $i>j$.

WLOG $(f(a_n,b_n))_n$ is constant. If 1, $\mathcal{M} \models \varphi(a_i,b_j)$ iff $i \le j$.

If 0, let $b_i' = b_{i+1}$. Then $\mathcal{M} \models \varphi(a_i, b_j')$ iff $i \le j$.

Case 2: $\forall n$, $\hat{f}_{b_n}(p) = 1$ and $\hat{f}^{a_n}(q) = 0$

Get$^{\text{WLOG}}$ $\mathcal{M} \models \varphi(a_i, b_j)$ iff $i \ge j$. Fix $k \ge 1$. For $i,j \le k$, $a_i'' = a_{k-i}$, $b_i'' = b_{k-i}$

Then $\mathcal{M} \models \varphi(a_i'', b_j'')$ iff $i \le j$. This suffices for OP by ES3 #6.

NOW

We build $(a_n)_{n\ge0}$ from $A = M^x$ and $(b_n)_{n \ge 0}$ from $B = M^y$ st $\forall n$

1. $\hat{f}_{b_n}(p) \ne \hat{f}^{a_n}(q)$
2. $\forall i<n$, $f(a_n, b_i) = \hat{f}_{b_i}(p)$
3. $\forall i<n$, $f(a_i, b_n) = \hat{f}^{a_i}(q)$.

Pick a_0, b_0 st $\hat{f}_{b_0}(p) \ne \hat{f}^{a_0}(q)$ (Choose $U = X$, $V = Y$ in $\neg$(*))

Suppose we have $(a_i)_{i<n}$ and $(b_i)_{i<n}$. Pick a_n, b_n.

Let $U = \bigcap_{i<n} \{u \in X : \hat{f}_{b_i}(u) = \hat{f}_{b_i}(p)\}$ open nbhd of p.

Let $V = \bigcap_{i<n} \{v \in Y : \hat{f}^{a_i}(v) = \hat{f}^{a_i}(q)\}$ open nbhd of q.

By $\neg$(*) $\exists$ $a_n \in A \cap U$ and $b_n \in B \cap V$ st $\hat{f}_{b_n}(p) \ne \hat{f}^{a_n}(q)$.

So $\forall i<n$, $f(a_n, b_i) = \hat{f}_{b_i}(a_n) = \hat{f}_{b_i}(p)$

$f(a_i, b_n) = \hat{f}^{a_i}(b_n) = \hat{f}^{a_i}(q)$

$\square$

FTS3 $\Rightarrow$ FTS2.

<u>Remark 19.1</u> The previous proof did not use Hausdorffness of X or Y.

Repeat proof with coarser topology on Y generated $[\varphi(a,y)]$ for all $a \in M^x$.

Given $p \in S_x(M)$, get φ-definition $\psi(y)$ to be a Boolean combination of $\varphi(a,y)$ for $a \in M^x$.

<u>Remark 19.2</u> T complete, $\mathcal{L}$ ctble.

1) $\aleph_0$-stable $\Rightarrow$ κ-stable $\forall$ $\kappa \geq \aleph_0$. (Thm 17.2)

2) stable $\rightarrow$ κ-stable $\forall$ $\kappa^{\aleph_0} = \kappa$. (see proof of FTS2 $\Rightarrow$ FTS1]

<u>Shelah 1976</u> Let $\Sigma(T) = \{\kappa : T \text{ is } \kappa\text{-stable}\}$. Then $\Sigma(T)$ is one of:

- $\{\kappa \geq \aleph_0\}$ $\leftarrow$ ω-stable — superstable — stable — ACF_p, TFDAG, InfSets
- $\{\kappa \geq 2^{\aleph_0}\}$ — superstable — stable — $Th(\mathbb{Z}, +, 0)$
- $\{\kappa^{\aleph_0} = \kappa\}$ $\leftarrow$ strictly stable — stable — $Th(\mathbb{Z}^\omega, +, 0)$, separably closed field, that is not ACF
- $\emptyset$ $\leftarrow$ unstable — RG, DLO

<u>Motto</u>: Stable algebraic structures are "nice".

<u>Ex</u>: (fields) $T = Th(F)$ where F is a field

[Macintyre[1971] / Cherlin-Shelah[1980]] If T is superstable then $F \models ACF$.

[Macintyre-Shelah-Wood 1975] If F is separably closed then T is stable.

<u>Stable Fields Conjecture</u> If T is stable then F is separably closed.

<u>Open</u>: Is $Th(\mathbb{C}(t))$ stable?