Part III Model Theory, Lecture 20, 25 Nov

Stable Groups

Def 20.1 Let $\mathcal{L}$ be a language. An $\mathcal{L}$-structure G is an *expansion of a group* if $\mathcal{L}$ contains the language of groups and the reduct of G to the group language is a group.

We conflate G with its universe.

Examples

① If G is an abelian group then $Th(G,+,0)$ is stable (folklore).

② If G is a free group then $Th(G,\cdot,1)$ is stable (Sela 2006)

③ If $P=\{2^n : n\geq 1\}$ then $Th(\mathbb{Z},+,0,P)$ is stable (Moosa–Scanlon 2007)

④ Let G be an algebraic group over some $K\models ACF$. Consider expansion of G by relation symbols for all subsets of G^n $(n\geq 1)$ definable in the field language. Then $Th(G)$ is ω-stable.

Let $T=Th(G)$ where G is an expansion of a group. Given an $\mathcal{L}_G$-formula $\varphi(x)$, let $\varphi(G)=\{a\in G : G\models\varphi(a)\}$. Recall that $X\subseteq G$ is *definable* if $X=\varphi(G)$ for some $\mathcal{L}_G$-formula $\varphi(x)$.

Def 20.2 Let $\varphi(x,y_1,\dots,y_n)$ be an $\mathcal{L}$-formula. Define $\mathcal{H}_\varphi$ to be the collection of all finite-index subgroups of G of the form $\varphi(G,\bar b)$ for some $\bar b\in G^n$.

Let $G^\circ(\varphi)=\bigcap\mathcal{H}_\varphi$ $\left(:=\bigcap_{H\in\mathcal{H}_\varphi}H\right)$. If $\mathcal{H}_\varphi=\emptyset$ then $G^\circ(\varphi)=G$.

Example 20.3 $G=(\mathbb{Z},+,\cdot,0,1)$ $\varphi(x,y)$ is $\exists z(x=y\cdot z)$ (y divides x)

$\varphi(\mathbb{Z},m)=m\mathbb{Z}$. So $G^\circ(\varphi)=\{0\}$.

$\left[m\mathbb{Z}\in\mathcal{H}_\varphi \ \forall\, m\neq 0 \qquad \{0\}\notin\mathcal{H}_\varphi\right]$ $\left[\bigcap_{m\neq 0}m\mathbb{Z}=\{0\}\right]$.

__Theorem 20.4__ (Baldwin–Saxl 1976)

Assume $T = Th(G)$ is stable. Then for any $\mathcal{L}$-formula $\varphi(x, y_1, \dots, y_n)$, there is a finite $\mathcal{F} \subseteq \mathcal{H}_\varphi$ st $G^0(\varphi) = \bigcap \mathcal{F}$. Moreover, $G^0(\varphi)$ is definable by an $\mathcal{L}$-formula.

__Proof__ Fix $\varphi(x, \bar{y})$. WLOG $\mathcal{H}_\varphi \neq \emptyset$.

__Claim 1__ $\exists\, m \geq 1$ st $\forall$ finite $\mathcal{H} \subseteq \mathcal{H}_\varphi$ then $\exists\, \mathcal{F} \subseteq \mathcal{H}$, $|\mathcal{F}| = m$, st $\bigcap \mathcal{H} = \bigcap \mathcal{F}$.

__Proof__: Suppose not. Fix $m \geq 1$. Then $\exists$ finite $\mathcal{H} \subseteq \mathcal{H}_\varphi$ st $\forall\, \mathcal{F} \subseteq \mathcal{H}$, $|\mathcal{F}| = m$, we have $\bigcap \mathcal{H} \subsetneq \bigcap \mathcal{F}$. After thinning $\mathcal{H}$ we may assume $|\mathcal{H}| > m$, and if $\mathcal{F} \subsetneq \mathcal{H}$ then $\bigcap \mathcal{H} \subsetneq \bigcap \mathcal{F}$. Let $\mathcal{H} = \{H_1, \dots, H_k\}$ $(k > m)$.

For $1 \leq i \leq k$, choose $g_i \in \left(\bigcap_{j \neq i} H_j\right) \setminus H_i$. Given $I \subseteq \{1, \dots, k\}$, let

$g_I = \prod_{i \in I} g_i$. Then $g_I \in H_j$ iff $j \notin I$:

$$j \notin I \Rightarrow g_i \in H_j \;\forall\, i \in I \Rightarrow g_I \in H_j.$$

$$j \in I \Rightarrow g_{I \setminus \{j\}} \in H_j \text{ and } g_j \notin H_j \Rightarrow g_I \notin H_j.$$

Choose $\bar{b}_j \in G^n$ st $H_j = \varphi(G, \bar{b}_j)$. Let $a_i = g_{<i}$ $\left(= \prod_{j<i} g_j\right)$

Then $G \models \varphi(a_i, \bar{b}_j)$ iff $g_{<i} \in H_j$ iff $i \leq j$.

Since k can be chosen arb. large, we get OP for $\varphi(x, \bar{y})$ by ES3 #6. //

Fix $m \geq 1$ as in Claim 1. Let $\psi(x, \bar{y}_1, \dots, \bar{y}_m)$ be $\bigwedge_{i=1}^{m} \varphi(x, \bar{y}_i)$.

__Claim 2__: $\mathcal{H}_\psi$ contains a minimal element.

__Proof__: Suppose not. There is $H_0 \supsetneq H_1 \supsetneq H_2 \supsetneq \dots$ with $H_i \in \mathcal{H}_\psi$.

Choose $g_i \in H_i \setminus H_{i+1}$. Then $g_i \in H_j$ iff $j \leq i$. So ψ has OP w.r.t. $\not{\!\!\text{⚡}}$. //

Let H be a minimal element of $\mathcal{H}_\psi$. Note $H = \bigcap \mathcal{F}$ where $\mathcal{F} \subseteq \mathcal{H}_\varphi$, $|\mathcal{F}| = m$.

Claim 3: $H = G^0(\varphi)$.

Proof: $G^0(\varphi) \leq H$ by def. of $G^0(\varphi)$. Fix $K \in \mathcal{H}_\varphi$. WTS $H \leq K$.

By Claim 1, $H \cap K \in \mathcal{H}_\varphi$. By minimality of H, $H = H \cap K$, i.e. $H \leq K$. //

Claim 4: $G^0(\varphi)$ is definable by an $\mathcal{L}$-formula.

Proof: Let $k = [G : G^0(\varphi)]$. Then $a \in G^0(\varphi)$ iff $\forall\, \bar{b} \in G^n$,

if $\varphi(G, \bar{b})$ is a subgroup of G of index at most k then $\varphi(a, \bar{b})$.

This expressible by an $\mathcal{L}$-formula. //

$\square$