Part III Model Theory, Lecture 21, 27 Nov

Setting: G expansion of a group. $T = Th(G)$. Assume T is stable.

Def 21.1: Let G° be the intersection of all definable finite index subgroups of G.

Note: $G^\circ = \bigcap_\varphi G^\circ(\varphi)$

Example 21.2

① $Th(\mathbb{Z}, +, 0) = T$. If $G \models T$ then $G^\circ = \bigcap_{n \geq 1} nG$. So if $G = \mathbb{Z}$, $G^\circ = \{0\}$

If G is $\aleph_0$-saturated then G° is nontrivial.

② G is an algebraic group over some ACF. G° is the connected component of G wrt the Zariski topology.

Remark 21.3: G° is a normal subgroup.

Goal: Another description of G°.

Def 21.4: $X \subseteq G$ is bi-generic if $\exists\, a_1, \dots, a_n, b_1, \dots, b_n \in G$ st $G = \bigcup_{i=1}^n a_i X b_i$.

Lemma 21.5 If $X \subseteq G$ is definable then X or $G \setminus X$ is bi-generic.

Proof: Suppose not. We build $(a_i)_{i \geq 1}$, $(b_i)_{i \geq 1}$ st $a_i b_j \in X$ iff $i \leq j$.
[So if $\varphi(x)$ defines X then $\varphi(x \cdot y)$ has OP.] Let $a_0 \in X$ and $b_0 = 1$.

Choose $a_n \notin \bigcup_{i<n} X b_i^{-1}$ and $b_n \notin \bigcup_{i \leq n} a_i^{-1}(G \setminus X)$

$a_n b_i \notin X \;\forall i < n$ and $a_i b_n \in X \;\forall i \leq n$ ☆ □

Example 21.6 $Th(\mathbb{Z}, +, <, 0)$ is unstable. $\mathbb{N}$ and $\mathbb{Z} \setminus \mathbb{N}$ are definable and not bi-generic.

Lemma 21.7 If $X, Y \subseteq G$ are definable and $X \cup Y$ is bi-generic, then X or Y is bi-generic.

Proof: Assume $G = \bigcup_{i=1}^n a_i (X \cup Y) b_i = \bigcup_{i=1}^n a_i X b_i \cup \bigcup_{i=1}^n a_i Y b_i$.

If A is bi-generic then so is X. $\qquad \underbrace{\qquad\qquad}_{A} \qquad \underbrace{\qquad\qquad}_{B}$

Suppose A is not bi-generic. Then $G \setminus A$ is bi-generic by 21.5.

So B is bi-generic (since $G \setminus A \subseteq B$). So Y is bi-generic. $\square$

<u>Def 21.8</u> $p \in S_1(G)$ is <u>bi-generic</u> if every $X \in p$ is bi-generic.

<u>Convention</u>: Identify $\mathcal{L}_G$-formula $\varphi(x)$ with $\varphi(G)$.

<u>Proposition 21.9</u> There is a bi-generic type $p \in S_1(G)$.

<u>Proof</u>: Let $q = \{\neg X : X \subseteq G$ is definable and not bi-generic$\}$. Then q is fin. sat. in G:

Fix $X_1, \ldots, X_n \subseteq G$ definable, not bi-generic. Then $X_1 \cup \ldots \cup X_n$ is not bi-generic by 21.7

So $\neg X_1 \cap \ldots \cap \neg X_n = \neg(X_1 \cup \ldots \cup X_n) \neq \emptyset$. Extend q to some $p \in S_1(G)$.

If $X \in p$ then $\neg X \notin q$, so X is bi-generic. $\square$

<u>Def 21.10</u> Given $p \in S_1(G)$ and $g \in G$, define $gp = \{gX : X \in p\}$.

<u>ES4</u>: $gp \in S_1(G)$.

<u>Def 21.11</u>: Given $p \in S_1(G)$ and $X \subseteq G$ definable, let

$$H_X^p = \{g \in G : \forall a \in G,\ aX \in p \text{ iff } aX \in gp\}$$

(T stable.)

<u>Theorem 21.12</u> If $p \in S_1(G)$ and $X \subseteq G$ is definable, then H_X^p is a definable subgroup of G. Moreover, if p is bi-generic then H_X^p has finite index.

<u>Proof</u>: Fix $p \in S_1(G)$ and definable $X \subseteq G$. Exc: H_X^p is a subgroup.

Since p is definable $\exists\ \psi(y)$ st $\forall a \in G$, $aX \in p$ iff $G \models \psi(a)$

[$\psi(y)$ is def. for p wrt the formula $\varphi(x,y)$ given by "$y^{-1} \cdot x \in X$".]

So $g \in H_X^p$ iff $G \models \forall a\,(\psi(a) \leftrightarrow \psi(g^{-1}a))$.

Given $q, r \in S_1(G)$, write $q \sim r$ iff $\forall a,b \in G$, $aXb \in q$ iff $aXb \in r$.

<u>Main Claim</u>: There are only finitely many bi-generic types in $S_1(G)$ mod $\sim$.

<u>Proof</u>: Next time.

Assume p is bi-generic. Suppose $\exists\, g_1, g_2, g_3, \ldots$ in G st $g_i^{-1}g_j \notin H_X^p$ $\forall\, i \neq j$.

So $\forall\, i \neq j$ $\exists\, a \in G$ st $aX \in p$ iff $aX \notin g_i^{-1}g_j p$, ie.

$$g_i aX \in g_i p \text{ iff } g_i aX \notin g_j p.$$

So $\forall\, i \neq j$, $g_i p \nsim g_j p$. But each $g_i p$ is bi-generic [ES4], contradicting the Main Claim. $\square$

<u>Corollary 21.13</u> If $p \in S_1(G)$ is bi-generic then

$$G^0 = \bigcap_{\substack{X \subseteq G \\ \text{def.}}} H_X^p$$

So $G^0 = \{g \in G : gp = p\} =: \mathrm{Stab}(p)$.

<u>Proof</u>: $G^0 \leq H_X^p$ $\forall$ def. $X \subseteq G$. For the other direction, it suffices to fix definable finite index normal subgroup K of G, and show $\bigcap_{X \subseteq G} H_X^p \leq K$.

Check: $H_K^p = K$. $\square$