Part III Model Theory, Lecture 22, 30 Nov

G expansion of a group. $T = Th(G)$ stable. Fix definable $X \subseteq G$.

Given $p, q \in S_1(G)$, write $p \sim q$ iff $\forall a, b \in G$, $aXb \in p$ iff $aXb \in q$.

Lemma 22.1 There are only finitely many bi-generic types in $S_1(G)$ modulo $\sim$.

Proof (non-examinable)

Let $\theta(x)$ be a formula defining X. WLOG $\theta(x)$ is an $\mathcal{L}$-formula. (add constants)

Let $\varphi(x; y, z)$ be $\theta(y^{-1} \cdot x \cdot z^{-1})$. Note $\varphi(x; a, b)$ defines aXb.

Given $M \models T$ and $p, q \in S_1(M)$, write $p \sim q$ if $\forall a, b \in M$, $\varphi(x; a, b) \in p$ iff $\varphi(x; a, b) \in q$. Given a set q of $\mathcal{L}_M$-formulas in free vble x, we define

$R(q) \in \mathbb{N} \cup \{-1, \infty\}$.

$R(q) \geq 0$ iff q is a type wrt M.

$R(q) \geq n+1$ iff $\exists\, N \succeq M$ and $p_i \in S_1(N)$ for $i \geq 0$, st $p_i \supseteq q$, $R(p_i) \geq n$, and $p_i \not\sim p_j$ $\forall\, i \neq j$.

Note: If $p \subseteq q$ then $R(q) \leq R(p)$

Strategy: 1. $R(x = x) < \infty$.

2. $\#\{\text{max rank } p \in S_1(G)\}/\sim$ is finite.

3. Any bi-generic $p \in S_1(G)$ has max rank.

Write $\varphi(x; y, z)$ as $\varphi(x; v)$ (where $v = (y, z)$).

Given $M \models T$ and $q(x)$ and a cardinal λ (possibly finite), define

$$\Gamma(q, \lambda) = \bigcup_{\sigma \in \mathbb{N}^\lambda} q(x_\sigma) \cup$$
$$\left\{ \varphi(\underline{x_\sigma}, \underline{v_{s,i,j}}) \leftrightarrow \neg\varphi(\underline{x_\tau}, \underline{v_{s,i,j}}) : \begin{array}{l} \sigma, \tau \in \mathbb{N}^\lambda,\ s \in \mathbb{N}^{<\lambda},\ i, j \in \mathbb{N}, \\ i \neq j,\ si \leq \sigma,\ sj \leq \tau \end{array} \right\}$$

Claim 1: $R(q) \geq n$ iff $\Gamma(q, n)$ is consistent.

Proof: (see notes)

Claim 2: $R(x=x) = n < \infty$. (for some n).

Proof: Suppose $R(x=x) = \infty$. By Claim 1, $\Gamma(x=x, n)$ is consistent $\forall n \geq 1$.
By Compactness, $\Gamma(x=x, \lambda)$ is consistent for any λ. Fix $\kappa \geq |L| + \aleph_0$. WTS: T is not κ-stable.
Choose minimal λ st $\kappa < 2^\lambda$. Let $(a_\sigma, b_{s,i,j}) \models \Gamma(x=x, \lambda)$ in some $N \models T$.
Choose $M \preceq N$ st $b_{s,i,j} \in M$ $\forall$ s,i,j and $|M| \leq |N^{<\lambda}| \leq \kappa$.
Let $p_\sigma = tp(a_\sigma / M)$. If $\sigma \neq \tau$ then $\exists$ s,i,j st $i \neq j$, $si \leq \sigma$, $sj \leq \tau$,
and so $\varphi(x, b_{s,i,j}) \in p_\sigma$ iff $\neg\varphi(x, b_{s,i,j}) \notin p_\tau$. So $|S_1(M)| \geq |N^\lambda| = 2^\lambda > \kappa$. //

Claim 3: There are only finitely many rank n types in $S_1(G)$ mod $\sim$.

Proof: Otherwise $R(x=x) \geq n+1$. //

Claim 4: For any $M \models T$ and any $p(x)$ over M, $\exists$ finite $q \subseteq p$ st $R(p) = R(q)$.

Proof: Suppose $R(p) = n$. Then $\Gamma(p, n+1)$ is inconsistent by Claim 3. So $\exists$ finite $q \subseteq p$
st $\Gamma(q, n+1)$ is inconsistent. So $R(q) \leq n$. So $n = R(p) \leq R(q) \leq n$. //

Claim 5: Given $M \models T$ and L_M-formulas $\psi_1(x), \psi_2(x)$, we have

$$R(\psi_1(x) \vee \psi_2(x)) = \max\{R(\psi_1(x)), R(\psi_2(x))\}.$$

Proof: ETS $\leq$. By pigeonhole and since the types p_i in the def. of R are complete. //

Claim 6: Given $M \models T$, $a, b \in M$, and $q(x)$, we have $R(aqb) = R(q)$.

Proof: ETS $\geq$. By induction. Suppose $R(q) \geq n+1$. Fix $N \succeq M$ and $p_i \in S_1(N)$
witnessing it. Let $q_i = ap_ib$. Then $q_i \supseteq aqb$ and $R(q_i) \geq n$ by induction.
Fix $i \neq j$. $\exists$ $c, d \in N$ st $\varphi(x, c, d) \in p_i$ iff $\neg\varphi(x, c, d) \in p_j$.

So $\varphi(x; ac, db) \in q_i$ iff $\neg\varphi(x, ac, db) \in q_j$. So $q_i \neq q_j$. So $R(aqb) \geq n+1$. //

Claim 7: If $p \in S_1(G)$ is bi-generic, then $R(p) = n$.

Proof: By Claim 4, $\exists\ Y \in p$ st $R(Y) = R(p)$. So Y is bi-generic.

Since $R(G) = R(x=x) = n$, $\exists\ a, b \in G$ st $R(aYb) = n$ by Claim 5.

So $R(Y) = n$ by Claim 6. So $R(p) = n$. //

Claims 3 + 7 yields the result. $\square$

More to say about the "structure" of the collection of bi-generic types in $S(G)$.