Part III Model Theory, Lecture 23, 2 Dec

Def 23.1 A group G is *amenable* if there is a left-invariant finitely additive probability measure on the subsets of G.

Ex: finite groups, solvable groups, fin. gen. groups of "subexponential growth".

Non-ex: nonabelian free groups, $SL_3(\mathbb{R})$ [Banach–Tarski]

Let $\mathcal{L}$ be a language and let G is an $\mathcal{L}$-structure expanding a group.

Def 23.2: G is *definably amenable* if there is a left inv. fin. add. prob. measure on the definable subsets of G.

Theorem 23.3 [Newelski–Petrykowski 2006]
If $Th(G)$ is stable then G is definably amenable.

Proof
By Prop 21.9, there is a bi-generic type $p \in S_1(G)$. Given a definable set $X \subseteq G$

let $H_X = H_X^p = \{g \in G : \forall a \in G,\ aX \in p \text{ iff } aX \in gp\}$.

and $D_X = \{g \in G : X \in gp\}$.

By Theorem 21.12, H_X is a definable finite-index subgroup of G.

Claim: D_X is a union of left cosets of H_X.

Pf: Fix $a \in D_X$. Fix $g \in aH_X$. WTS $g \in D_X$, ie. $X \in gp$.

We have $X \in ap$ and $a^{-1}g \in H_X$. So $a^{-1}X \in p \Rightarrow a^{-1}X \in a^{-1}gp \Rightarrow X \in gp$. //

Notation: Given $H \leq G$ and $D \subseteq G$, a union of left cosets of H, let

$|D/H|$ is the number of left cosets of H contained in D.

Given definable set $X \subseteq G$, let $\mu(X) = \dfrac{|D_X / H_X|}{|G/H_X|} \in [0,1]$.

μ is left-invariant: Fix X and $c \in G$.

Then $H_{cX} = \{ g \in G : \forall a \in G,\ acX \in p \text{ iff } acX \in gp \} = H_X$

Also $g \in D_{cX}$ iff $cX \in gp$ iff $X \in c^{-1}gp$ iff $c^{-1}g \in D_X$ iff $g \in cD_X$

So $D_{cX} = cD_X$

So $\mu(cX) = \dfrac{|D_{cX}/H_{cX}|}{|G/H_{cX}|} = \dfrac{|cD_X/H_X|}{|G/H_X|} = \dfrac{|D_X/H_X|}{|G/H_X|} = \mu(X).$

$\mu(G) = 1$: $H_G = G = D_G$

μ is finitely additive: Fix disjoint definable $X, Y \subseteq G$

WTS: $\mu(X \cup Y) = \mu(X) + \mu(Y)$

Claim: $D_{X \cup Y} = D_X \cup D_Y$

Pf: $X \cup Y \in gp$ iff $X \in gp$ or $Y \in gp$.

Claim: $D_X \cap D_Y = D_{X \cap Y} = D_\emptyset = \emptyset$

Claim: $H_X \cap H_Y \leq H_{X \cup Y}$

Pf: Fix $g \in H_X \cap H_Y$. For any $a \in G$

$a(X \cup Y) \in p$ iff $aX \in p$ or $aY \in p$ iff $aX \in gp$ or $aY \in gp$ iff $a(X \cup Y) \in gp$ //

Note: Suppose $K \leq H \leq G$ and $[G:H] = n$, $[H:K] = m$

Suppose $D \subseteq G$ is a union of l cosets of H. Then D is a union of lm cosets of K.

and $\dfrac{|D/K|}{|G/K|} = \dfrac{lm}{nm} = \dfrac{l}{n} = \dfrac{|D/H|}{|G/H|}$

Now let $K = H_X \cap H_Y$. Then

$\mu(X \cup Y) = \dfrac{|D_{X \cup Y}/H_{X \cup Y}|}{|G/H_{X \cup Y}|} = \dfrac{|D_{X \cup Y}/K|}{|G/K|} = \dfrac{|D_X \cup D_Y/K|}{|G/K|}$

$$= \frac{|D_X/K| + |D_Y/K|}{|G/K|} = \frac{|D_X/K|}{|G/K|} + \frac{|D_Y/K|}{|G/K|} = \frac{|D_X/H_X|}{|G/H_X|} + \frac{|D_Y/H_Y|}{|G/H_Y|}$$

$$= \mu(X) + \mu(Y).$$ $\square$

<u>Remark 23.4</u>

1) $\mu(X) = \mu(D_X)$. In fact, one can show $\mu(X \triangle D_X) = 0$

Moreover, μ is the unique left-inv. finitely additive prob. measure on definable subsets of G.

E.g. $(\mathbb{Z}, +, <, 0)$ $\forall r \in [0,1]$ $\exists \mu$ st $\mu(\mathbb{N}) = r$.

OTOH: If $X \subseteq \mathbb{Z}$ is definable in a stable expansion of $(\mathbb{Z}, +)$, then there is a set D, which is a union of cosets of a nontrivial subgroup of $\mathbb{Z}$, st $X \triangle D$ has upper Banach density 0.

2) G stable group. Let $\Gamma = \{p \in S_1(G) : p \text{ is bi-generic}\}$.

Then Γ is closed. Fix $p, q \in \Gamma$. Let $G_1 \succeq G$ and $b \in G_1$, $b \models q$.

Let $p_1 \in S_1(G_1)$ be the "definable extension" of p, ie. $\varphi(x,c) \in p_1$ ← any $\mathcal{L}_{G_1}$-form.

iff $G_1 \models \psi(c)$ where $\psi(y)$ is φ-def. of p.

Let $G_2 \succeq G_1$ and $a \in G_2$, $a \models p_1$. Then let $p * q = tp(ab/G) \in S_1(G)$

<u>Fact</u>: $(\Gamma, *)$ is a compact Hausdorff group.

If G is sufficiently saturated, then $\Gamma \cong G/G^0$. Fix $p \in \Gamma$.

For $X \subseteq G$ definable, $\mu(X) = \mathrm{Haar}(\{aG^0 : X \in ap\})$

well-defined: $G^0 = \mathrm{Stab}(p)$.