Part III - Model Theory - Review

6) a) $\mathcal{M} \models$ any universal consequence of T

Show $\mathcal{M}$ embeds in a model of T.

WTS: $T \cup \mathrm{Diag}(\mathcal{M})$ has a model.

Suppose not. Then $\exists$ a $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$
quantifier-free

st $T \cup \{\varphi(a_1, \ldots, a_n)\}$ is inconsistent.

So $T \models \neg \exists \bar{x} \varphi(\bar{x})$

$T \models \underbrace{\forall \bar{x} \neg \varphi(\bar{x})}_{\text{universal sentence}}$ (Generalization)

So $\mathcal{M} \models \forall \bar{x} \neg \varphi(\bar{x})$, $\lightning$.

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b) $\mathcal{M}$ is a model of any existential consequence of T.

WTS: $\exists$ $\underline{\mathcal{N} \models T}$ that embeds in an elementary extension of $\mathcal{M}$.

<u>Goal</u>: Find $\underline{\mathcal{N}} \models T$ st $\mathrm{Th}_M(\mathcal{M}) \cup \mathrm{Diag}(\mathcal{N})$ ← new constants.
?

<u>Step 1</u> Get right $\mathcal{N}$.

Let $\Sigma = \{\neg \exists \bar{x} \psi(\bar{x}) : \mathcal{M} \models \neg \exists \bar{x} \psi(\bar{x});\ \psi(\bar{x})$ q.f. $\mathcal{L}$-formula $\}$.

WTS: $T \cup \Sigma$ has a model.

Fix $\psi_1(\bar{x}), \ldots, \psi_n(\bar{x})$ q.f. st $\neg \exists \bar{x} \psi_i(\bar{x}) \in \Sigma$ $\forall$ i.

WTS $T \cup \{\neg \exists \bar{x} \psi_1(\bar{x}), \ldots, \neg \exists \bar{x} \psi_n(\bar{x})\}$ is satisfiable.

Suppose not. Then $T \models \exists \bar{x} \psi_1(\bar{x}) \vee \ldots \vee \exists \bar{x} \psi_n(\bar{x})$

i.e. $T \models \exists \bar{x} (\psi_1(\bar{x}) \vee \ldots \vee \psi_n(\bar{x}))$.

So $\mathcal{M} \models \exists \bar{x} (\psi_1(\bar{x}) \vee \ldots \vee \psi_n(\bar{x}))$.

So $\mathcal{M} \models \exists \bar{x}\, \psi_1(\bar{x}) \vee \dots \vee \psi_n(\bar{x})$

So $\mathcal{M} \models \exists \bar{x}\, \psi_i(\bar{x})$ for some $1 \le i \le n$.

But $\mathcal{M} \models \neg \exists \bar{x}\, \psi_i(\bar{x})$ by def. of Σ. $\lightning$

Let $\mathcal{N} \models T \cup \Sigma$.

Step 2 Show $Th_{\mathcal{M}}(\mathcal{M}) \cup Diag(\mathcal{N})$ is satisfiable/(consistent)

Fix a q.f. $\mathcal{L}$-formula $\psi(x_1, \dots, x_m)$ and $c_1, \dots, c_m \in N$ st

$\psi(c_1, \dots, c_m) \in Diag(\mathcal{N})$

WTS: $Th_{\mathcal{M}}(\mathcal{M}) \cup \{\psi(c_1, \dots, c_m)\}$ is consistent.

If not, then $Th_{\mathcal{M}}(\mathcal{M}) \models \neg \exists \bar{x}\, \psi(\bar{x})$.

ie $\mathcal{M} \models \neg \exists \bar{x}\, \psi(\bar{x})$

So $\neg \exists \bar{x}\, \psi(\bar{x}) \in \Sigma$ ☆

So $\mathcal{N} \models \neg \exists \bar{x}\, \psi(\bar{x})$

But $\mathcal{N} \models \psi(\bar{c})$. $\lightning$

like part (a)

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10) $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \preceq \mathcal{N}$

Suppose $p \in S_n(\mathcal{M})$ is definable

$\left[\text{ie. } \forall\ \mathcal{L}\text{-formula } \varphi(x_1, \dots, x_n, \bar{y})\ \exists\ \mathcal{L}_M\text{-formula } \underline{\psi(\bar{y})} \text{ st } \forall\ \bar{b} \in M^{\bar{y}},\ \varphi(\bar{x}, \bar{b}) \in p \text{ iff } \mathcal{M} \models \psi(\bar{b}).\right]$

WTS: $\exists$ a unique $q \in S_n(\mathcal{N})$ st q extends p and q is <u>definable over M</u>

Proof: Given $\varphi(\bar{x}, \bar{y})$ let $d\varphi(\bar{y})$ be the $\mathcal{L}_M$-formula that gives a φ-definition for p.

Define q (set of $\mathcal{L}_N$-formulas in $\bar{x}$) st given $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in N^{\bar{y}}$

$\varphi(\bar{x}, \bar{b}) \in q$ iff $\mathcal{N} \models d\varphi(\bar{b})$.

* q is complete since $\mathcal{N} \models \neg d\varphi(\bar{b})$ or $\mathcal{N} \models d\varphi(\bar{b})$
* q is consistent. ETS: every formula in q is consistent. (check: $d(\varphi \wedge \psi)(\bar{y}) \equiv d\varphi(\bar{y}) \wedge d\psi(\bar{y})$)

  Fix $\varphi(\bar{x}, \bar{b}) \in q$. Note: $\mathcal{M} \models \forall \bar{y}\, (d\varphi(\bar{y}) \rightarrow \exists \bar{x}\, \varphi(\bar{x}, \bar{y}))$.

  So $\mathcal{N}$ satisfies this. Since $\mathcal{N} \models d\varphi(\bar{b})$, we know $\varphi(\bar{x}, \bar{b})$ is consistent.

We have $q \in S_n(\mathcal{N})$.

* q is definable over $\mathcal{M}$ by construction.
* q extends p: Fix $\varphi(\bar{x}, \bar{b})$ with $\bar{b} \in \mathcal{M}$

  Then $\varphi(\bar{x}, \bar{b}) \in p$ iff $\mathcal{M} \models d\varphi(\bar{b})$
  iff $\mathcal{N} \models d\varphi(\bar{b})$
  iff $\varphi(\bar{x}, \bar{b}) \in q$.

* q is unique: Suppose $q' \in S_n(\mathcal{N})$ is another extension of p that is definable over $\mathcal{M}$. WTS $q = q'$. Fix $\varphi(\bar{x}, \bar{y})$. We show $d\varphi(\bar{y})$ is a φ-definition for q' (and so $q = q'$). Let $\psi(\bar{y})$ be an $\mathcal{L}_M$-formula which defines q' wrt $\varphi(\bar{x}, \bar{y})$. Given $\bar{b} \in M^{\bar{y}}$ we have

$$\mathcal{M} \models \psi(\bar{b}) \text{ iff } \varphi(\bar{x}, \bar{b}) \in q' \text{ iff } \varphi(\bar{x}, \bar{b}) \in q \text{ iff } \mathcal{M} \models d\varphi(\bar{b}).$$

So $\mathcal{M} \models \forall \bar{y}\, (d\varphi(\bar{y}) \leftrightarrow \psi(\bar{y}))$

So $\mathcal{N}$ satisfies this.

8) Prove T is $\aleph_0$-cat. iff $S_n(T)$ is finite $\forall\ n \geq 1$.

($\Longrightarrow$): Suppose $\exists\, n$ st $S_n(T)$ is infinite.

Since $S_n(T)$ is compact Hausdorff, $\exists$ non-isolated $p \in S_n(T)$.

By OTT $\exists$ ctble model $\mathcal{M} \models T$ omitting p.

Also $\exists$ ctble $\mathcal{N} \models T$ realizing p. So $\mathcal{M} \not\cong \mathcal{N}$.

($\Longleftarrow$): Assume $S_n(T)$ is finite $\forall\ n \geq 1$. Then all types over ϕ are isolated.

Fix ctble $\mathcal{M}, \mathcal{N} \models T$. Say $\mathcal{M} = \{a_n : n \geq 0\}$ and $\mathcal{N} = \{b_n : n \geq 0\}$

Build partial elementary $(f_n)_{n \geq 0}$ st $\operatorname{dom}(f_n) \subseteq \mathcal{M}$ is fin't $\forall\ n$

and $a_n \in \operatorname{dom}(f_{n+1})$, $b_n \in \operatorname{Im}(f_{n+1})$ $\forall\ n$.

Then $f = \bigcup f_n$ will be isomorphism from $\mathcal{M}$ to $\mathcal{N}$.

Let $f_0 = \phi$ (elementary since $\mathcal{M}, \mathcal{N} \models T$)

Given f_n. Enumerate $\operatorname{dom}(f_n) = \{c_1, \dots, c_m\}$. Let $\Theta(y_1, \dots, y_m, x)$

isolate $tp(c_1, \dots, c_m, a_n/\phi) \in S_{m+1}(T)$.

$\mathcal{M} \models \exists x\, \Theta(\bar{c}, x)$. So $\mathcal{N} \models \exists x\, \Theta(f_n(\bar{c}), x)$

So pick $d \in \mathcal{N}$ st $\mathcal{N} \models \Theta(f_n(\bar{c}), d)$.

So $\Theta(\bar{y}, x) \in tp(f_n(\bar{c}), d/\phi)$

$\left([\Theta(\bar{y}, x)] = \{ \quad \} \right)$

So $tp(f_n(\bar{c}), d/\phi) = tp(\bar{c}, a_n/\phi)$

So $f_n \cup \{(a_n, d)\}$ is partial elementary.

Similarly find $e \in \mathcal{M}$ st $\underbrace{f_n \cup \{(a_n, d), (e, b_n)\}}_{f_{n+1}}$ is partial

elementary. (Consider formula isolating $tp(f_n(\bar{c}), d, b_n/\phi)$)

2) Notation $N \models T$, $\varphi(\bar{x}, \bar{b})$, $\varphi(N, \bar{b}) := \{\bar{a} \in N^{\bar{x}} : N \models \varphi(\bar{a}, \bar{b})\}$.

a) $M \preceq N$. Fix $\varphi(\bar{x}, \bar{y})$. (length n, length m) Fix $\bar{b} \in N^{\bar{y}}$. Suppose $\varphi(N, \bar{b})$ is defined by an $\mathcal{L}_M$-formula $\psi(\bar{x})$

WTS $\exists\, \bar{c} \in M^{\bar{y}}$ st $\varphi(N, \bar{c}) = \varphi(N, \bar{b})$

$N \models \exists \bar{y} (\forall \bar{x} (\varphi(\bar{x}, \bar{y}) \leftrightarrow \psi(\bar{x})))$.

$M \models$ "

Choose $\bar{c} \in M^{\bar{y}}$ st $M \models \forall \bar{x} (\varphi(\bar{x}, \bar{c}) \leftrightarrow \psi(\bar{x}))$

So $N \models$ "

So $N \models \forall \bar{x} (\varphi(\bar{x}, \bar{b}) \leftrightarrow \varphi(\bar{x}, \bar{c}))$.

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b) Fix $\varphi(\bar{x}, \bar{y})$. Assume only finitely many sets of the form $\varphi(N, \bar{b})$ for $\bar{b} \in N^{\bar{y}}$. Fix $M \preceq N$.

WTS. $\forall\, \bar{b} \in N^{\bar{y}}$, $\varphi(N, \bar{b})$ is definable using an $\mathcal{L}_M$-formula.

There is $k \geq 1$, st

$N \models \exists \bar{y}_1 \dots \bar{y}_k \forall \bar{y} \bigvee_{i=1}^{k} \forall \bar{x} (\varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{y}_i))$

So $M \models$ " " " "

So $\exists\, \bar{b}_1, \dots, \bar{b}_k \in M^{\bar{y}}$ st

$M \models \forall \bar{y} \bigvee_{i=1}^{k} \forall \bar{x} (\varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{b}_i))$

So $N \models$ " " "

So $\forall\, \bar{b} \in N^{\bar{y}}$ $\exists\, 1 \leq i \leq k$ st $\varphi(N, \bar{b}) = \varphi(N, \bar{b}_i)$

c) Suppose $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{N}$ is $|M|^+$-saturated.

Suppose any $\varphi(\mathcal{N}, \bar{b})$ definable using an $\mathcal{L}_M$-formula. ☆

WTS: Only finitely many sets of the form $\varphi(\mathcal{N}, \bar{b})$.

<u>Proof</u>: Suppose $\exists$ infinitely many such sets. ☆☆ Consider a partial type in $\bar{y}$

$$\Gamma(\bar{y}) = \{ \exists \bar{x} (\varphi(\bar{x}, \bar{y}) \leftrightarrow \neg \varphi(\bar{x}, \bar{b})) : \bar{b} \in M^{\bar{0}} \}.$$

WTS $\Gamma(\bar{y})$ is realized in $\mathcal{N}$

If $\bar{c} \models \Gamma$ then $\varphi(\mathcal{N}, \bar{c}) \neq \varphi(\mathcal{N}, \bar{b})$ $\forall\, \bar{b} \in M^{\bar{0}}$ ye do ☆

By saturation, ETS $\Gamma(\bar{y})$ is finitely satisfiable in $\mathcal{N}$.

But this follows from ☆☆. $\square$

<u>ADDED AFTER CLASS</u>

1) ($\Rightarrow$): Immediate from definition of $\mathcal{M} \preceq \mathcal{N}$.

($\Leftarrow$): WTS: For any $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and $\bar{a} \in M^n$, $\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{N} \models \varphi(\bar{a})$.

<u>Induction on formulas</u>: If $\varphi(\bar{x})$ is atomic then result follows since $\mathcal{M} \subseteq \mathcal{N}$.

Conjunctions + negations are easy.

Assume the result for $\varphi(x_1, \dots, x_n, y)$ and consider $\exists y\, \varphi(\bar{x}, y)$. Fix $\bar{a} \in M^n$.

If $\mathcal{N} \models \exists y\, \varphi(\bar{a}, y)$ then $\mathcal{M} \models \exists y\, \varphi(\bar{a}, y)$ by our main assumption.

Conversely, $\mathcal{M} \models \exists y\, \varphi(\bar{a}, y) \Rightarrow \mathcal{M} \models \varphi(\bar{a}, b)$ for some $b \in M$,

$\Rightarrow \mathcal{N} \models \varphi(\bar{a}, b)$ for some $b \in M$ (by induction)

$\Rightarrow \mathcal{N} \models \varphi(\bar{a}, b)$ for some $b \in N$

$\Rightarrow \mathcal{N} \models \exists y\, \varphi(\bar{a}, y)$.

3) First note that each $\mathcal{M}_i$ is a graph.

WTS: $\exists\, i$ st $\mathcal{M}_i$ is a Rado graph. Suppose not. Then for all i, $\exists$ finite disjoint $A_i, B_i \subseteq M_i$ st $\nexists\, c \in M_i$ with $R^{\mathcal{M}_i}(a,c)$ $\forall\, a \in A_i$ and $\neg R^{\mathcal{M}_i}(b,c)$ $\forall\, b \in B_i$. Let $A = A_1 \cup \ldots \cup A_n$, $B = B_1 \cup \ldots \cup B_n$. So A, B are finite and $A \cap B = \emptyset$ since $A_i \cap B_i = \emptyset$ $\forall\, i$ and $M_i \cap M_j = \emptyset$ $\forall\, i \neq j$. Since $\mathcal{M} \models RG$, $\exists\, c \in M$ st $R^{\mathcal{M}}(a,c)$ $\forall\, a \in A$ and $\neg R^{\mathcal{M}}(b,c)$ $\forall\, b \in B$. Fix $i \leq n$ st $c \in M_i$. Since $\mathcal{M}_i \subseteq \mathcal{M}$, we have $R^{\mathcal{M}_i}(a,c)$ $\forall\, a \in A_i$ and $\neg R^{\mathcal{M}_i}(b,c)$ $\forall\, b \in B_i$. $\lightning$

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4) Assume T has QE and fix $\mathcal{M}, \mathcal{N} \models T$ with $\mathcal{M} \subseteq \mathcal{N}$. WTS $\mathcal{M} \preceq \mathcal{N}$. Fix an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and $\bar{a} \in M^n$. WTS $\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{N} \models \varphi(\bar{a})$. Choose quantifier-free $\psi(\bar{x})$ st

$$T \models \forall \bar{x}\, (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Then $\mathcal{M} \models \varphi(\bar{a}) \underset{\mathcal{M} \models T}{\overset{\uparrow}{\Leftrightarrow}} \mathcal{M} \models \psi(\bar{a}) \underset{\mathcal{M} \subseteq \mathcal{N}}{\overset{\uparrow}{\Leftrightarrow}} \mathcal{N} \models \psi(\bar{a}) \underset{\mathcal{N} \models T.}{\overset{\uparrow}{\Leftrightarrow}} \mathcal{N} \models \varphi(\bar{a})$.

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5) a) If $p \in S_n(T)$ is non-isolated then $\exists$ ctble $\mathcal{M} \models T$ omitting p.

b) Fix $q = \{\psi_1(\bar{x}), \ldots, \psi_k(\bar{x})\} \subseteq p$ (each ψ_i is an $\mathcal{L}_A$-formula). Fix $\mathcal{N} \succeq \mathcal{M}$ and $\bar{a} \in N^n$ st $\bar{a} \models p$. Then $\mathcal{N} \models \psi_i(\bar{a})$ $\forall\, i$. So $\mathcal{N} \models \exists \bar{x} \bigwedge_{i=1}^{k} \psi_i(\bar{x})$. So $\mathcal{M} \models \exists \bar{x} \bigwedge_{i=1}^{k} \psi_i(\bar{x})$, as desired.

c) Suppose $p \in S_n^{\mathcal{M}}(A)$ is isolated. Fix an $\mathcal{L}_A$-formula $\varphi(\bar{x})$ isolating p. Then $\exists\, \bar{a} \in M^n$ st $\mathcal{M} \models \varphi(\bar{a})$ (by part (b)). So $tp^{\mathcal{M}}(\bar{a}/A) \in S_n^{\mathcal{M}}(A)$ contains $\varphi(\bar{x})$, which implies $tp^{\mathcal{M}}(\bar{a}/A) = p$ since φ isolates p. So $\bar{a} \models p$.

d) Suppose $\mathcal{M} \models T$ is prime. WTS $\mathcal{M}$ realizes no non-isolated types in $S_n^{\mathcal{M}}(\emptyset) = S_n(T)$. So suppose $p \in S_n(T)$ is non-isolated. By OTT, $\exists\, \mathcal{N} \models T$ omitting p. Since $\mathcal{M}$ is prime, we may assume $\mathcal{M} \preceq \mathcal{N}$. So p is also omitted in $\mathcal{M}$.

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7) [Note: I will omit details. This would be a long problem for an exam.]

Claim: T is a complete theory with QE. ← (needed later)

Proof: It suffices to prove this for any reduct of T to a finite sublanguage. I.e., we may assume that for some fixed $n \geq 1$, T is the theory of DLO + $c_0 < c_1 < \ldots < c_{n-1}$.

Now fix a finite $\mathcal{L}$-structure $\mathcal{A}$, which embeds in some model of T, and show $T \cup Diag(\mathcal{A})$ is complete. (This gives QE by our characterization from lecture, and completeness of T when $\mathcal{A} = \emptyset$)

To show $T \cup Diag(\mathcal{A})$ is complete, we use Vaught's Test. In particular, we show $T \cup Diag(\mathcal{A})$ is $\aleph_0$-categorical (note this theory has no finite models). Fix ctble $\mathcal{M}, \mathcal{N} \models T \cup Diag(\mathcal{A})$. WLOG $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{A} \subseteq \mathcal{N}$. Let f_0 be the identity on $\mathcal{A}$. Then f_0 is a partial embedding from $\mathcal{M}$ to $\mathcal{N}$ with finite domains. Now do usual back and forth between $\mathcal{M}$ and $\mathcal{N}$ starting with f_0 (note that $c_i^{\mathcal{M}} = c_i^{\mathcal{N}} = c_i^{\mathcal{A}} \in A \ \forall i$ so we don't need to worry about the constants. //

Claim: T has 3 countable models.

Proof: See lecture notes. The models are $\mathcal{M}_1$ st (c_n) unbounded, $\mathcal{M}_2$ st (c_n) bounded but w/o supremum, and $\mathcal{M}_3$ st (c_n) has supremum. //

Claim: $\mathcal{M}_1$ is the prime model.

Proof: Fix $\mathcal{N} \models T$. WTS: $\exists$ an elementary embedding from $\mathcal{M}_1$ to $\mathcal{N}$. By QE, it suffices to show $\exists$ an embedding from $\mathcal{M}_1$ to $\mathcal{N}$. Let $M_1 = \{c_n^{\mathcal{M}_1} : n \geq 0\} \cup \{a_n : n \geq 0\}$. We build $(f_n)_{n\geq 0}$ partial embeddings from $\mathcal{M}_1$ to $\mathcal{N}$ st $dom(f_n) = \{c_k^{\mathcal{M}_1} : k \geq 0\} \cup \{a_k : k < n\}$. Then $f = \bigcup f_n$ is desired embedding.

Start with $f_0 : c_n^{\mathcal{M}_1} \mapsto c_n^{\mathcal{N}} \ \forall n$. Given f_n, let $Im(f_n) = \{c_k^{\mathcal{N}} : k \geq 0\} \cup \{b_k : k < n\}$ where $f_n(a_k) = b_k$. Choose ℓ st $c_\ell^{\mathcal{M}_1} > a_k \ \forall k \leq n$. Then $b_k < c_\ell^{\mathcal{N}} \ \forall k < \ell$. Choose $b_n \in N$ satisfying the same order type among $c_0^{\mathcal{N}}, \dots, c_\ell^{\mathcal{N}}, b_0, \dots, b_{n-1}$ as a_n does among $c_0^{\mathcal{M}_1}, \dots, c_\ell^{\mathcal{M}_1}, a_0, \dots, a_{n-1}$ (just like in back + forth for DLO). Then set $f_{n+1} = f_n \cup \{(a_n, b_n)\}$.

Claim: $\mathcal{M}_2$ is saturated.

Proof: Fix finite $A \subseteq M_2$ and $p \in S_1^{\mathcal{M}_2}(A)$. WTS p is realized in $\mathcal{M}_2$. By QE, p is completely determined by the induced cut in $A \cup \{c_n^{\mathcal{M}_2} : n \geq 0\}$.

Case 1: p contains $x < c_n$ for some $n \geq 0$. Then p is determined by a cut in the finite set $A \cup \{c_0^{\mathcal{M}_3}, \dots, c_n^{\mathcal{M}_3}\}$. But all such cuts are realized in $\mathcal{M}_3$ as in the case of DLO.

Case 2: p contains $x > c_n$ for all $n \geq 0$. Let $A_0 = \{a \in A : a > c_n^{\mathcal{M}_3} \ \forall n\}$.

Case 2a: p contains $x \geq a$ for some $a \in A_0$. Then p determined by cut in the finite set A_0, hence is realized.

Case 2b: p contains $x < a$ for all $a \in A_0$. So p is determined by: $x > c_n \ \forall n$ and $x < a \ \forall a \in A_0$. Since $(c_n^{\mathcal{M}_3})$ has no sup, there is some $b \in M_3$ realizing this cut.

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9) Fix $p \in S_1(M)$. WTS: p is definable wrt $\varphi(x,y)$.

Case 1: $\forall a \in M$, $\varphi(x,a) \notin p$. Then $y \neq y$ is a φ-definition for p.

Case 2: $\exists a \in M$, $\varphi(x,a) \in p$. We show $\varphi(a,y)$ is a φ-definition for p.

Fix $b \in M$. If $\varphi(x,b) \in p$ then $\{\varphi(x,a), \varphi(x,b)\}$ is consistent and so $\mathcal{M} \models \varphi(a,b)$ since φ is an equivalence relation. Conversely, if $\varphi(a,b)$ holds then $\varphi(x,b) \in p$ since $\{\varphi(x,b), \neg\varphi(x,a)\}$ is inconsistent.

Remark: One can show that $\varphi(x,y)$ is stable. So this exercise also follows from FTS (overkill).