1) a) Fix $g, h \in H_X^p$. We first show $g^{-1} \in H_X^p$. Observe: $g^{-1} \in H_X^p \iff$
$\forall a \in G,\ aX \in p$ iff $aX \in g^{-1}p \iff \forall a \in G,\ aX \in p$ iff $gaX \in p$
$\iff \forall b \in G,\ g^{-1}bX \in p$ iff $bX \in p \iff \forall b \in G,\ bX \in gp$ iff $bX \in p$
$\iff g \in H_X^p$.

Now we show $gh \in H_X^p$, i.e., $\forall a \in G$, $aX \in p$ iff $aX \in g^{-1}hp$. So fix $a \in G$. Then
$aX \in p$ iff $aX \in g^{-1}p$ iff $gaX \in p$ iff $gaX \in hp$ iff $aX \in g^{-1}hp$.

b) [See lecture notes]

c)* Suppose X is bi-generic. Then $\exists\ a_1, \dots, a_n \in G$ st $G = \bigcup_{i=1}^{n} (a_i + X)$. Since $G \in p$ and p is a type, $\exists\ 1 \le i \le n$ st $a_i + X \in p$. Let $a = a_i$. Now fix $g \in H_X^p$. Since $a + X \in p$, we have $a + X \in g + p$, i.e. $(a-g)+X \in p$. So $(a+X) \cap ((a-g)+X) \in p$. Since p is a type, we have $(a+X) \cap ((a-g)+X) \neq \emptyset$. Fix $x, y \in X$ st $a + x = a - g + y$. Then $g = y - x \in X - X$.

*__Note__: I switched to additive notation for this part of the problem.

---

2) For any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, we have some $\mathcal{L}_N$-formula $d\varphi(\bar{y})$ st $\forall\ \bar{b} \in N^{\bar{y}}$, $\varphi(\bar{x}, \bar{b}) \in p$ iff $N \models d\varphi(\bar{b})$. Since the number of $\varphi(\bar{x}, \bar{y})$ is $|\mathcal{L}| + |\aleph_0|$, and each $d\varphi(\bar{y})$ uses only finitely many parameters from N, $\exists\ A \subseteq N$ st $|A| \le |\mathcal{L}| + |\aleph_0|$ and A contains all parameters used in all $d\varphi(\bar{y})$'s. By ES1 #9, $\exists\ M \preceq N$ st $|M| \le |\mathcal{L}| + \aleph_0$ and $A \subseteq M$. Then p is definable over M by construction.

---

3) a) Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}_M$-formula and fix $\bar{b} \in N^{\bar{y}}$ st $\varphi(\bar{x}, \bar{b}) \in p$. Write $\varphi(\bar{x}, \bar{y})$ as $\psi(\bar{x}, \bar{y}, \bar{d})$ where $\psi(\bar{x}, \bar{y}, \bar{z})$ is an $\mathcal{L}$-formula and $\bar{d} \in M^{\bar{z}}$. Let $d\varphi(\bar{y}, \bar{z})$ be an $\mathcal{L}_M$-formula defining p wrt $\psi(\bar{x}, \bar{y}, \bar{z})$. Then $N \models d\varphi(\bar{b}, \bar{d})$, and so $N \models \exists \bar{y}\, d\varphi(\bar{y}, \bar{d})$. So $M \models \exists \bar{y}\, d\varphi(\bar{y}, \bar{d})$. Pick $\bar{c} \in M^{\bar{y}}$ st $M \models d\varphi(\bar{c}, \bar{d})$. Then $N \models d\varphi(\bar{c}, \bar{d})$. So $\psi(\bar{x}, \bar{c}, \bar{d}) \in p$, i.e., $\varphi(\bar{x}, \bar{c}) \in p$.

b) Suppose not. Choose $\varphi(\bar{x}, \bar{b}) \in p$ st $N \models \neg\varphi(\bar{a}, \bar{b})\ \forall\ \bar{a} \in M^{\bar{x}}$.

We define (by induction) sequences $(\bar{a}_i)_{i=0}^{\infty}$ from $M^{\bar{x}}$ and $(\bar{b}_i)_{i=0}^{\infty}$ from $M^{\bar{y}}$ st

(i) $\varphi(\bar{x}, \bar{b}_i) \in p\ \forall\ i \ge 0$, and

(ii) $M \models \varphi(\bar{a}_i, \bar{b}_j)$ iff $i \ge j$.

In particular, (ii) contradicts stability of T. Fix $k \geq 0$ and suppose $(\bar{a}_i, \bar{b}_i)_{i < k}$ have been found satisfying (i) and (ii). Consider the $\mathcal{L}_M$-formula $\psi(\bar{x}, \bar{y})$ given by

$$\bigwedge_{i<k} \varphi(\bar{x}, \bar{b}_i) \wedge \bigwedge_{i<k} \neg\varphi(\bar{a}_i, \bar{y}) \wedge \varphi(\bar{x}, \bar{y})$$

Then $\psi(\bar{x}, \bar{b}) \in p$ by (i) and since $\mathcal{N} \models \neg\varphi(\bar{a}_i, \bar{b})$ $\forall$ $i<k$ (by assumption).

By part (a), there is $\bar{b}_k \in M^{\bar{y}}$ st $\psi(\bar{x}, \bar{b}_k) \in p$. In particular, $\varphi(\bar{x}, \bar{b}_k) \in p$.

We also have $\mathcal{M} \models \neg\varphi(\bar{a}_i, \bar{b}_k)$ $\forall$ $i<k$. Since p is consistent, so is $\psi(\bar{x}, \bar{b}_k)$, and so there is $\bar{a}_k \in M^{\bar{x}}$ st $\mathcal{M} \models \psi(\bar{a}_k, \bar{b}_k)$. In particular, we have $\mathcal{M} \models \varphi(\bar{a}_k, \bar{b}_i)$ $\forall$ $i \leq k$. So $(\bar{a}_i, \bar{b}_i)_{i \leq k}$ satisfy (i) and (ii).

c) Let T be DLO. Choose $\mathcal{M} = \mathbb{Q}$ and let $\mathcal{N} \succeq \mathcal{M}$ be $\aleph_1$-saturated. Let $p \in S_1(N)$ be st $x > a$ is in p for all $a \in N$.

<u>Claim</u>: p is definable over $\mathcal{M}$.

<u>Pf</u>: By QE it suffices to consider $x = y$ and $x > y$. Given $a \in N$,

$$x = a \in p \quad \text{iff} \quad \mathcal{N} \models a \neq a$$
$$x > a \in p \quad \text{iff} \quad \mathcal{N} \models a = a$$

So we can use $y \neq y$ and $y = y$ as our defining formulas

(We've really shown that p is "definable over $\emptyset$.") //

Now, since $\mathcal{N}$ is $\aleph_1$-saturated, $\exists$ $b \in N$ st $b > a$ $\forall$ $a \in M$.

Then $x > b$ is in p, but $\nexists$ $a \in M$ st $\mathcal{N} \models a > b$.

---

4) Fix $\mathcal{M} \models T$ and $p \in S_n(\mathcal{M})$. We show p is definable (and so T is stable by FTS). Fix an $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$. Let $\mathcal{N} \succeq \mathcal{M}$ be st there is $\bar{a} \in N^{\bar{x}}$ st $\bar{a} \models p$.

Set $X = \{\bar{b} \in N^{\bar{y}} : \mathcal{N} \models \varphi(\bar{a}, \bar{b})\}$. By assumption, there is an $\mathcal{L}_M$-formula $\psi(\bar{y})$ which defines $X \cap M^{\bar{y}}$ in $\mathcal{M}$. Given $\bar{b} \in M^{\bar{y}}$, we have $\mathcal{M} \models \psi(\bar{b})$ iff $\bar{b} \in X \cap M^{\bar{y}}$ iff $\bar{b} \in X$ iff $\mathcal{N} \models \varphi(\bar{a}, \bar{b})$ iff $\varphi(\bar{x}, \bar{b}) \in p$.