Part III - Model Theory
Types in $\text{Th}(\mathbb{Z},+,0,1)$.

Let $T=\text{Th}(\mathbb{Z},+,0,1)$. As stated in lecture, T has quantifier elimination after adding infinitely many binary relations $(\equiv_n)_{n\geq 2}$ for congruence modulo n (i.e., $x\equiv_n y$ if and only if $\exists z(x-y=nz)$).

Now fix $\mathcal{M}\models T$. Let $\mathbb{P}$ be the set of primes. During lecture, we fixed a function $f:\mathbb{P}\to\mathbb{N}$ such that $0\leq f(n)<n$ for all $n\in\mathbb{P}$, and we defined the type

$$p_f:=\{x\neq a : a\in M\}\cup\{x\equiv_n f(n) : n\in\mathbb{P}\}.$$

Then I incorrectly claimed that this determines a complete 1-type over M. But this is wrong because p_f does not determine residue classes modulo all *prime powers*. So really all we've done is shown $S_1(M)\geq|M|+2^{\aleph_0}$.

However, the main point still stands that to complete a non-realized type over M, we just need to specify all possible congruence relations, and there are at most $2^{\aleph_0}$ many ways to do this. So a correct way to handle this is as follows.

Let $\mathfrak{P}$ be the set of all prime powers, and let F be the set of functions $f\colon\mathfrak{P}\to\mathbb{N}$ such that:

(*i*) $0\leq n<f(n)$ for all $n\in\mathfrak{P}$, and

(*ii*) for any finite set $A\subseteq\mathfrak{P}$ there is an integer $m\in\mathbb{Z}$ such that $m\equiv_n f(n)$ for all $n\in A$.

Now, for $f\in F$, define

$$q_f=\{x\neq a : a\in M\}\cup\{x\equiv_n f(n) : n\in\mathfrak{P}\}.$$

Then q_f *really does* determine a complete 1-type in $S_1(M)$ since now we've fully determined all possible residue classes, and we've done so in a consistent manner. In conclusion,

$$S_1(M)=\{\text{tp}(a/M) : a\in M\}\cup\{q_f : f\in F\},$$

and we still see that $|S_1(M)|=|M|+2^{\aleph_0}$.

Note that if $f\in F$ then q_f extends the type p_{f_0} defined above, where $f_0=f{\restriction}\mathbb{P}$. In other words, given a function $f\colon\mathbb{P}\to\mathbb{N}$ with $0\leq f(n)<n$ for all $n\in\mathbb{P}$, if we let X_f be the set of functions in F that extend f, then $\{q_h : h\in X_f\}$ gives us all of the completions of p_f to a type in $S_1(M)$.

Those of you familiar with the profinite completion of the group of integers may see some resemblance here. This is not a coincidence, and I may have time to say something more about this later on when we talk about stable groups more generally. In the mean time, you can try to come up with a more intrinsic description of the set F.