Part III - Model Theory
Comments on chains

Let $\mathcal{L}$ be a language. The following definition was given in lecture.

Definition 3.7. Let α be a limit ordinal. A collection $(\mathcal{M}_i)_{i<\alpha}$ of $\mathcal{L}$-structures is a **chain** if for all $i < \alpha$, $\mathcal{M}_i \subseteq \mathcal{M}_{i+1}$ and $\mathcal{M}_i \subseteq \mathcal{M}_\beta$ for any limit ordinal β with $i < \beta < \alpha$.

This definition was made overly complicated because I first stated it incorrectly. A better way to state this is as follows.

Definition 3.7 (alternate). Let α be a limit ordinal. A collection $(\mathcal{M}_i)_{i<\alpha}$ of $\mathcal{L}$-structures is a **chain** if $\mathcal{M}_i \subseteq \mathcal{M}_j$ for all $i < j < \alpha$.

Similarly, $(\mathcal{M}_i)_{i<\alpha}$ is an **elementary chain** (according to the definition from lecture) if and only if $\mathcal{M}_i \preceq \mathcal{M}_j$ for all $i < j < \alpha$.

You can treat it as an exercise to prove the equivalence of these definitions. The basic point is that $\subseteq$ and $\preceq$ are transitive relations on $\mathcal{L}$-structures, which is a good thing to be aware of anyway.

A final comment is that the previous notions make sense for chains indexed by arbitrary linear orders. In other words, one need not assume the index set is a well-order with no maximal element. But we will focus mainly on ordinal-indexed chains.