The Compactness Theorem

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We work with a fixed language $\mathcal{L}$. Recall the Compactness Theorem.

The Compactness Theorem. *An $\mathcal{L}$-theory T is satisfiable if and only if it is finitely satisfiable.*

In these notes, I discuss the "original proof" of Compactness via Gödel's Completeness Theorem. Nothing below is examinable, and none of the definitions introduced below will be used anywhere in the course. But since the Compactness Theorem is one of the most important tools in model theory, it will be helpful to have some understanding of where it comes from.

The idea behind Gödel's Completeness Theorem is to reconcile the notion of first-order satisfaction (or "truth") in $\mathcal{L}$-structures, with the idea of formal syntactic proofs. In particular, given a set Σ of $\mathcal{L}$-formulas, and some other $\mathcal{L}$-formula φ, one can rigorously defined the notion "Σ proves φ", written $\Sigma \vdash \varphi$, as follows.

Definition 1. $\Sigma \vdash \varphi$ if there is a finite sequence $\psi_1, \ldots, \psi_n$ of $\mathcal{L}$-formulas such that ψ_n is φ and, for all $1 \leq k \leq n$, one of the following holds:

1. φ_k is in Σ (i.e., φ_k is an "assumption");
2. φ_k is in a fixed set of *logical axioms* (discussed below);
3. there are $i, j < k$ such that φ_j is the formula $\varphi_i \to \varphi_k$ (i.e., φ_k can be obtained from previous formulas in the sequence via *modus ponens*).

In the previous definition, *modus ponens* is the logical rule of inference that "from A and $A \to B$, one can infer B". This a meta-principal of logical reasoning. By contrast, the set of *logical axioms*, referred to in the second part of the definition, is a specific set of $\mathcal{L}$-formulas. I omit a full description of this set, to avoid getting bogged down in non-critical details, but the idea is that the logical axioms are an explicitly defined collection of statements that are true in *every* structure. Examples of logical axioms include things like $(\varphi \wedge \psi) \to \varphi$ and $\forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$, where φ and ψ are arbitrary formulas.

The previous setup can vary from source to source. Some authors prefer to add more rules of inference in order to make the set of logical axioms more efficient.

While the above setup applies to arbitrary formulas, let us now focus on sentences for simplicity. We have two notions of what it can mean for a sentence φ to be a "logical consequence" of an $\mathcal{L}$-theory T:

1. $T \models \varphi$, i.e., for any $\mathcal{L}$-structure $\mathcal{M}$, if $\mathcal{M}$ satisfies all sentences in T, then $\mathcal{M}$ satisfies φ.
2. $T \vdash \varphi$, i.e., there is a formal proof of φ from the assumptions in T.

It turns out that these two notions are equivalent. This amazing fact was proved by Gödel in a pair of theorems now referred to as *Soundness* and *Completeness*.

The Soundness Theorem. *If $T \vdash \varphi$ then $T \models \varphi$.*

The Soundness Theorem is not very hard to prove. One just has to verify that satisfaction in first-order structures respects *modus ponens* (i.e., $\{\varphi, (\varphi \to \psi)\} \models \psi$), and that if φ is a logical axiom then $\mathcal{M} \models \varphi$ for any $\mathcal{L}$-structure $\mathcal{M}$. One should think of the Soundness Theorem as a sort of "temperature check" for the proof system. It tells us that we haven't accidentally included any suspicious or problematic logical axioms.

The real work lies in the converse implication.

The Completeness Theorem. *If $T \models \varphi$ then $T \vdash \varphi$.*

This result is call the Completeness Theorem because it tells us that our proof system is complete in the sense that anything "true" can be formally proved. Gödel's original proof from 1930 was highly technical, and these days one usually refers to a later proof by Leon Henkin from 1949. What Henkin did was show that if a set T of sentences is *consistent*, i.e., does not prove a contradiction $\varphi \wedge \neg\varphi$ for any sentence φ, then one can build a model of T using first-order symbols as a universe (modulo a certain equivalence relation). This kind of construction is now called a *Henkin construction*, and the proof requires the Axiom of Choice (via Zorn's Lemma) at a key step.

We can now return to the Compactness Theorem, which is a quick corollary of the above tools.

Proof of Compactness. If T is satisfiable then it is obvious finitely satisfiable. So suppose T is not satisfiable, i.e., there is no model $\mathcal{M}$ of T. Then we vacuously have $T \models \varphi \wedge \neg\varphi$, where φ is any fixed sentence. By the Completeness Theorem, $T \vdash \varphi \wedge \neg\varphi$. This brings us to the key observation that proofs are *finite*. In the formal proof of $\varphi \wedge \neg\varphi$ from T, only finitely many assumptions from T appear. So we have a finite set $T_0 \subseteq T$ such that $T_0 \vdash \varphi \wedge \neg\varphi$. By the Soundness Theorem, $T_0 \models \varphi \wedge \neg\varphi$, which implies that T_0 has no models. So we have found a finite subset of T that is unsatisfiable, as desired. □

Henkin's proof of the Completeness Theorem also implies the following useful result.

Downward Löwenheim-Skolem Theorem. *Let T be a satisfiable $\mathcal{L}$-theory. Then T has a model $\mathcal{M}$ with $|M| \leq |\mathcal{L}| + \aleph_0$.*

This statement follows from the fact that, in a Henkin construction, one builds a model whose underlying universe is $\mathcal{L}^*/\sim$, where $\mathcal{L}^*$ is obtained by adding at most $|\mathcal{L}|+\aleph_0$ new constant symbols to $\mathcal{L}$, and $\sim$ is a certain equivalence relation.

The Soundness and Completeness Theorems will not be used explicitly in this course. On the other hand, the Compactness Theorem and Downward Löwenheim-Skolem will be indispensable tools. (But, once again, the proofs of these results are non-examinable.)