Part III - Model Theory
Exam Review Sheet 2

These exercises are primarily focused on material around stability theory. You can (and should) assume the Fundamental Theorem of Stability in each problem.

1. Suppose G is a first-order expansion of a group, and $T = \mathrm{Th}(G)$. Fix a definable set $X \subseteq G$ and a type $p \in S_1(G)$. Recall that we set

$$H^p_X = \{g \in G : \text{ for all } a \in G,\ aX \in p \text{ iff } aX \in gp\}.$$

   (a) Show that H^p_X is a subgroup of G.

   (b) Suppose T is stable. Prove that H^p_X is definable.

   (c) Suppose G is abelian. Prove that if X is bi-generic then H^p_X is contained in $X - X := \{x - y : x, y \in X\}$.

2. Let $\mathcal{L}$ be a first-order language and suppose T is a complete $\mathcal{L}$-theory with infinite models. Suppose $\mathcal{N} \models T$ and $p \in S_n(\mathcal{N})$ is definable. Prove that there is some $\mathcal{M} \preceq \mathcal{N}$ such that p is definable over $\mathcal{M}$ and $|M| \leq |\mathcal{L}| + \aleph_0$.

   (See #10 on first exam review sheet for the definition of "definable over $\mathcal{M}$".)

3. Let $\mathcal{L}$ be a first-order language and suppose T is a complete $\mathcal{L}$-theory with infinite models. Fix a model $\mathcal{N} \models T$.

   (a) Suppose $p \in S_n(\mathcal{N})$ is definable over a model $\mathcal{M} \preceq \mathcal{N}$. Prove that for any $\mathcal{L}_M$-formula $\varphi(\bar{x}, \bar{y})$, if there is some $\bar{b} \in N^{\bar{y}}$ such that $\varphi(\bar{x}, \bar{b}) \in p$, then there is some $\bar{c} \in M^{|\bar{y}|}$ such that $\varphi(\bar{x}, \bar{c}) \in p$.

   (b) Assume T is stable. Suppose $p \in S_n(\mathcal{N})$ and p is definable over some $\mathcal{M} \preceq \mathcal{N}$. Prove that for any formula $\varphi(\bar{x}, \bar{b}) \in p$, there is some $\bar{a} \in M^{|\bar{x}|}$ such that $\mathcal{N} \models \varphi(\bar{a}, \bar{b})$.

   [Hint: Argue by contradiction and construct the order property for $\varphi(\bar{x}, \bar{y})$.]

   (c) Give an example to show that part (c) can fail if T is unstable.

4. Let $\mathcal{L}$ be a first-order language and suppose T is a complete $\mathcal{L}$-theory with infinite models. Given a model $\mathcal{M} \models T$ and a subset $X \subseteq M^n$ for some $n \geq 1$, we say that X is *definable in* $\mathcal{M}$ if there is an $\mathcal{L}_M$-formula $\varphi(x_1, \ldots, x_n)$ such that

$$X = \{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a})\}.$$

   Suppose that for any $\mathcal{N} \models T$ and any $X \subseteq N^n$ which is definable in $\mathcal{N}$, we have that for any $\mathcal{M} \preceq \mathcal{N}$, $X \cap M^n$ is definable in $\mathcal{M}$. Prove that T is stable.