Part III - Model Theory
Exam Review Sheet

This sheet includes modified examples from exams given in previous years, modified examples from our own classes and lectures, and some new examples.

1. Prove the *Tarski-Vaught Test:* Suppose $\mathcal{N}$ is an $\mathcal{L}$-structure and $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then $\mathcal{M} \preceq \mathcal{N}$ if and only if, for any $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n, y)$ and $a_1, \dots, a_n \in M$, if $\mathcal{N} \models \exists y \varphi(\bar{a}, y)$ then $\mathcal{M} \models \exists y \varphi(\bar{a}, y)$.

   (Similar to 2019 Exam, Question #1(a).)

2. Suppose T is a complete $\mathcal{L}$-theory and $\mathcal{N}$ is a model of T. Let $\varphi(x_1, \dots, x_n, y_1, \dots, y_n)$ be an $\mathcal{L}$-formula and, given $\bar{b} \in N^n$, let $\varphi(\mathcal{N}, \bar{b})$ denote the set $\{\bar{a} \in N^m : \mathcal{N} \models \varphi(\bar{a}, \bar{b})\}$.

   (a) Let $\mathcal{M} \preceq \mathcal{N}$. Prove that for any $\bar{b} \in N^n$, $\varphi(\mathcal{N}, \bar{b})$ is defined by an $\mathcal{L}_M$-formula if and only if there is some $\bar{c} \in M^n$ such that $\varphi(\mathcal{N}, \bar{b})$ is defined by $\varphi(\bar{x}, \bar{c})$.

   (b) Assume that there are only finitely many sets of the form $\varphi(\mathcal{N}, \bar{b})$ for $\bar{b} \in N^n$. Prove that, for any elementary substructure $\mathcal{M} \preceq \mathcal{N}$, any set of the form $\varphi(\mathcal{N}, \bar{b})$ for $\bar{b} \in N^n$ is definable using an $\mathcal{L}_M$-formula.

   (c) Suppose $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{N}$ is $|M|^+$-saturated. Assume any set of the form $\varphi(\mathcal{N}, \bar{b})$ for $\bar{b} \in N^n$ is definable using an $\mathcal{L}_M$-formula. Does it follow that there are only finitely many such sets?

   (Similar to 2019 Exam, Question #1(b, c).)

3. Let $\mathcal{M}$ be a model of the theory of RG of Rado graphs, and suppose we have a partition $M_1, \dots, M_n$ of M into a finite number of pairwise disjoint sets. Let $\mathcal{M}_i$ be the $\mathcal{L}$-substructure of $\mathcal{M}$ with universe M_i. Prove that there is some $1 \leq i \leq n$ such that $\mathcal{M}_i$ is a model of RG.

   (Similar to 2019 Exam, Question #2(b).)

4. An $\mathcal{L}$-theory T is called *model complete* if whenever $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} \preceq \mathcal{N}$. Prove that any theory with quantifier elimination is model complete.

   (Similar to 2015 Exam, Question #3(a).)

5. Let $\mathcal{L}$ be a countable language, and let T be a complete $\mathcal{L}$-theory with infinite models.

   (a) State the Omitting Types Theorem.

   (b) Suppose $\mathcal{M} \models T$, $A \subseteq M$, and $p \in S_n^{\mathcal{M}}(A)$. Prove that any finite subset $q \subseteq p$ is realized in M (viewing q as a partial type).

   (c) Suppose $\mathcal{M} \models T$ and $A \subseteq M$. Prove that every isolated type in $S_n^{\mathcal{M}}(A)$ is realized in $\mathcal{M}$.

   (d) Prove that any prime model of T is atomic.

(Similar to 2015 Exam, Question #5(a, c, d, f). Parts (b) and (e) are omitted since they are too easy.)

6. Let T be an $\mathcal{L}$-theory. An *universal* (resp., *existential*) $\mathcal{L}$-sentence is an $\mathcal{L}$-sentence of the form $\forall \bar{x} \varphi(\bar{x})$ (resp., $\exists \bar{x} \varphi(\bar{x})$), where $\varphi(\bar{x})$ is a quantifier-free $\mathcal{L}$-formula.

   (a) Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure such that $\mathcal{M} \models \psi$ for any universal $\mathcal{L}$-sentence ψ such that $T \models \psi$. Prove that $\mathcal{M}$ embeds in a model of T.

   (b) Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure such that $\mathcal{M} \models \psi$ for any existential $\mathcal{L}$-sentence ψ such that $T \models \psi$. Prove that there is a model of T that embeds in an elementary extension of $\mathcal{M}$.

   (Similar to 2002 Exam, Question #1(b, c).)

7. Let $\mathcal{L}$ be a language with a binary relation symbol $<$ and countably many constant symbols c_n for $n \geq 0$. Let T be the $\mathcal{L}$-theory asserting that $<$ is a DLO and $(c_n)_{n \geq 0}$ is a strictly increasing sequence. Prove that T is a complete $\mathcal{L}$-theory with exactly three countable models. Identify the prime model and countable saturated model of T.

   (Similar to 2002 Exam, Question 4(c).)

8. Let T be a complete $\mathcal{L}$-theory with infinite models. Prove that T is $\aleph_0$-categorical if and only if $S_n(T)$ is finite for all $n \geq 1$. (You may assume the Omitting Types Theorem, and basic topological properties of type spaces.)

   (Similar to 2002 Exam, Question #5.)

9. Suppose T is a complete theory and $\mathcal{M} \models T$. Let $\varphi(x, y)$ be a formula in two free variables that defines an equivalence relation on M. Prove that any type $p \in S_1(M)$ is definable with respect to $\varphi(x, y)$.

10. Suppose T is a complete theory with infinite models and fix models $\mathcal{M}, \mathcal{N} \models T$ with $\mathcal{M} \preceq \mathcal{N}$. Call a type $q \in S_n(\mathcal{N})$ *definable over* $\mathcal{M}$ if, for any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, q is definable with respect to $\varphi(\bar{x}, \bar{y})$ using an $\mathcal{L}_M$-formula. Now suppose $p \in S_n(\mathcal{M})$ is a definable type. Prove that for any $\mathcal{N} \succeq \mathcal{M}$, there is a unique type $q \in S_n(\mathcal{N})$ extending p that is definable over $\mathcal{M}$.