Part III - Model Theory
Examples Sheet 1 (revised 16 Oct)

Problems 3 and 9 should be submitted through the course Moodle by 11:59 pm on Wednesday, 28 October.

1. Suppose T is an $\mathcal{L}$-theory with arbitrarily large finite models. Show that T has an infinite model.

2. Let T be an $\mathcal{L}$-theory. Show that the following are equivalent.

   (i) T is complete.

   (ii) For any $\mathcal{L}$-sentence φ, if $T \not\models \varphi$ then $T \models \neg\varphi$.

   (iii) Any two models of T are elementarily equivalent.

3. Let $\mathcal{L}$ be the empty language. Show that the theory of infinite sets is κ-categorical for all $\kappa \geq \aleph_0$, and use this to prove that the theory of infinite sets is complete.

4. Let $\mathcal{L}$ be the language of groups. Write down a theory whose models are precisely nontrivial torsion-free divisible abelian groups, and prove that this theory is complete. (Recall that an abelian group G is *divisible* if for any $x \in G$ and $n > 0$ there is some $y \in G$ such that $x = ny$.)

5. Prove the *De Bruijn-Erdős Theorem*: If (V, E) is a graph and every finite subset of V is k-colorable, then V is k-colorable.

6. Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, with $M \subseteq N$. Show that $\mathcal{M} \subseteq \mathcal{N}$ if and only if, for any quantifier-free $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, we have

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n) \;\Leftrightarrow\; \mathcal{N} \models \varphi(a_1, \ldots, a_n).$$

7. Let $\mathcal{N}$ be an $\mathcal{L}$-structure and fix a nonempty set $A \subseteq \mathcal{N}$. Let $\mathcal{M}$ be the substructure of $\mathcal{N}$ generated by A. Show that for any $b \in M$ there is an $\mathcal{L}$-term $t(x_1, \ldots, x_n)$, and some $a_1, \ldots, a_n \in A$, such that $b = t^{\mathcal{M}}(a_1, \ldots, a_n)$.

8. Suppose α is a limit ordinal and $(\mathcal{M}_i)_{i<\alpha}$ is a chain of $\mathcal{L}$-structures.

   (a) Show that $\mathcal{N} = \bigcup_{i<\alpha} \mathcal{M}_i$ is a well-defined $\mathcal{L}$-structure, and $\mathcal{M}_i \subseteq \mathcal{N}$ for all $i < \alpha$.

   (b) Show that if $(\mathcal{M}_i)_{i<\alpha}$ is an elementary chain, then $\mathcal{M}_i \preceq \mathcal{N}$ for all $i < \alpha$.

9. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and fix $A \subseteq M$. Prove that there is an elementary substructure $\mathcal{N} \preceq \mathcal{M}$ such that $A \subseteq N$ and $|N| \leq |A| + |\mathcal{L}| + \aleph_0$.