Part III - Model Theory
Examples Sheet 2

Problems 3 and 8 should be submitted through the course Moodle by 11:59 pm on Wednesday, 11 November.

1. Prove that DLO is not κ-categorical for any uncountable κ.

2. Prove *Lemma 6.1*: Let T be an $\mathcal{L}$-theory such that, for any quantifier-free $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n, y)$, there is a quantifier-free $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ such that
$$T \models \forall \bar{x}(\exists y \varphi(\bar{x}, y) \leftrightarrow \psi(\bar{x})).$$
Then T has quantifier elimination.

3. Prove that the theory of nontrivial torsion-free divisible abelian groups (in the group language) has quantifier elimination.

4. Prove that DLO has quantifier elimination.

5. Let $\mathcal{L} = \{<\}$ be the language of orders.
   (*a*) Show that the $\mathcal{L}$-theory $\mathrm{Th}(\mathbb{Z}, <)$ does not have quantifier elimination.
   (*b*) Find a unary function $s\colon \mathbb{Z} \to \mathbb{Z}$ such that $y = s(x)$ is definable with an $\mathcal{L}$-formula, and the $(\mathcal{L} \cup \{s\})$-theory $\mathrm{Th}(\mathbb{Z}, <, s)$ has quantifier elimination.

6. Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure, $A \subseteq M$, and q is an n-type over A with respect to $\mathcal{M}$. Show that $q \subseteq p$ for some $p \in S_n^{\mathcal{M}}(A)$.

7. Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. Verify that the topology on $S_n^{\mathcal{M}}(A)$ defined in lecture is indeed a topology. Then show that a subset $C \subseteq S_n^{\mathcal{M}}(A)$ is clopen if and only if $C = [\varphi(\bar{x})]$ for some $\mathcal{L}_A$-formula $\varphi(\bar{x})$.

8. Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure. Show that $\mathcal{M}$ is κ-saturated if and only if for any $A \subseteq M$, with $|A| < \kappa$, every type in $S_1^{\mathcal{M}}(A)$ is realized in $\mathcal{M}$.