Part III - Model Theory
Examples Sheet 3

Problems 1 and 7 should be submitted through the course Moodle by 11:59 pm on Wednesday, 25 November.

1. Let T be a complete theory in a countable language.
   - (a) Show that any atomic model of T is $\aleph_0$-homogeneous.
   - (b) Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable $\aleph_0$-homogeneous models of T such that, for any $n \geq 1$ and $p \in S_n(T)$, $\mathcal{M}$ realizes p if and only if $\mathcal{N}$ realizes p. Prove that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.
   - (c) Show that any two prime models of T are isomorphic.
2. Fix $k \geq 1$ and let $\mathcal{L} = \{<, U_1, \ldots, U_k, c_0, c_1, c_2, \ldots\}$, where $<$ is a binary relation symbol, each U_i is a unary relation symbol, and each c_i is a constant symbol. Let T_k be the $\mathcal{L}$-theory asserting that $<$ is a dense linear order, $U_1, \ldots, U_k$ partition the universe into dense subsets, and, for all $n \geq 0$, $c_n < c_{n+1}$ and $U_1(c_n)$. Prove that T_k is complete, and that $I(T, \aleph_0) = k + 2$. (We proved this during lecture for $k = 1$.)
3. Suppose $\mathcal{M}$ is a κ-saturated $\mathcal{L}$-structure, and $A \subseteq M$ with $|A| < \kappa$. Show that $\mathcal{M}$ is κ-saturated as an $\mathcal{L}_A$-structure.
4. Let $\mathcal{M}$ and $\mathcal{N}$ be saturated $\mathcal{L}$-structures. Prove that if $\mathcal{M} \equiv \mathcal{N}$ and $|M| = |N|$, then $\mathcal{M} \cong \mathcal{N}$. (In light of Proposition 11.4, you may assume $\mathcal{M}$ and $\mathcal{N}$ are uncountable.)
5. Let T be a complete $\mathcal{L}$-theory with infinite models, and let $\kappa \geq |\mathcal{L}| + \aleph_0$. Show that T is κ-stable if and only if for any $\mathcal{M} \models T$ and $A \subseteq M$, if $|A| \leq \kappa$ then $|S_n(A)| \leq \kappa$ for all $n \geq 1$.
6. Let T be a complete $\mathcal{L}$-theory, and let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula. Show that $\varphi(\bar{x}, \bar{y})$ has the order property with respect to T if and only if for all $k \geq 1$,
$$T \models \exists \bar{x}_1 \ldots \exists \bar{x}_k \exists \bar{y}_1 \ldots \exists \bar{y}_k \left(\bigwedge_{i \leq j} \varphi(\bar{x}_i, \bar{y}_j) \wedge \bigwedge_{i > j} \neg\varphi(\bar{x}_i, \bar{y}_j) \right).$$
7. Let T be a complete $\mathcal{L}$-theory with quantifier elimination, and suppose $\mathcal{M} \models T$. Let $p \in S_n(M)$. Show that p is definable if and only if it is definable with respect to every atomic formula $\varphi(\bar{x}, \bar{y})$ with $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y}$ arbitrary.
8. Let $T = \mathrm{Th}(\mathbb{Z}, <)$. Show that every type in $S_n(\mathbb{Z})$ is definable.