Part III - Model Theory
Local infinity rank

Let T be a complete $\mathcal{L}$-theory and fix an $\mathcal{L}$-formula $\varphi(x,y)$, where x and y are tuples of variables. Given $\mathcal{M} \models T$ and $p,q \in S_x(M)$, we write $p \sim q$ if, for any $b \in M^y$, we have $\phi(x,b) \in p$ if and only if $\phi(x,b) \in q$. We now recall the definition of the rank R from lecture.

Definition 1. Let $\mathcal{M} \models T$ and let q be a set of $\mathcal{L}_M$-formulas in free variables x. The rank $R(q)$ takes values in $\mathbb{N} \cup \{\text{-}1, \infty\}$, and is defined inductively as follows.

- *(i)* $R(q) \geq 0$ if and only if q is finitely satisfiable in $\mathcal{M}$ (i.e., is actually a type over $\mathcal{M}$).
- *(ii)* Given $n \geq 0$, $R(q) \geq n+1$ if and only if there is some $\mathcal{N} \succeq \mathcal{M}$ and infinitely many pairwise $\sim$-inequivalent types $p_i \in S_x(N)$, for $i < \omega$, such that $q \subseteq p_i$ and $R(p_i) \geq n$.

Finally, define $R(q)$ to be the supremum of the set of n such that $R(q) \geq n$. If this set is $\mathbb{N}$, we let $R(q) = \infty$, and if this set is empty (i.e., q is inconsistent) then we let $R(q) = \text{-}1$.

Given q and $n \geq 0$, let $\Gamma(q,n)$ denote the following set of formulas:

$$\left(\bigcup_{\sigma \in \omega^n} q(x_\sigma)\right) \cup \{\varphi(x_\sigma, y_{s,i,j}) \leftrightarrow \neg\varphi(x_\tau, y_{s,i,j}) : \sigma, \tau \in \omega^n,\ s \in \omega^{<n},\ si \leq \sigma,\ sj \leq \tau,\ i \neq j\}$$

We now re-state Claim 1 from the proof of Lemma 22.1.

Proposition 2. *For any $n \geq 0$, $R(q) \geq n$ if and only if $\Gamma(q,n)$ is consistent (with T).*

Proof. The basic idea is that $\Gamma(q,n)$ codes up precisely the combinatorial information needed to witness $R(q) \geq n$. Other than dealing with notation, the only real difficulty comes from the fact that the types p_i in the definition of R are assumed to be complete. To overcome this we, first extend the relation $\sim$ to types that are possibly incomplete.

Given $\mathcal{M} \models T$ and (possibly partial) types $p(x), q(x)$ over $\mathcal{M}$, write $p \sim q$ if and only if there is no $b \in M^y$ such that $\phi(x,b) \in p$ and $\neg\phi(x,b) \in q$, or $\neg\phi(x,b) \in p$ and $\phi(x,b) \in q$. Define another rank R^* exactly as in Definition 1, except that the extensions p_i are allowed to be partial types over $\mathcal{N}$. Now the strategy for proving the proposition is to first show that the statement holds with R^* instead of R, and then show that R and R^* are the same.

Step 1: Given $\mathcal{M} \models T$ and q over $\mathcal{M}$, we have $R^*(q) \geq n$ if and only if $\Gamma(q,n)$ is consistent.

(There is not much content to this step, but it is a little bit technical. It might be better for one to write it out in their own way. The idea is that if we have infinitely many ω-branching trees of height n, then we can "shift indices" and join them together into a single ω-branching tree of height $n+1$.)

Proof of Step 1: We proceed by induction on n. For the base case $n = 0$, note that $\Gamma(q,0)$ is just q. So assume the result for n. We consider $n+1$.

Suppose $\Gamma(q,n{+}1)$ is consistent. Then it is satisfied in $\mathcal{N} \succeq \mathcal{M}$ by $(a_\sigma, b_{s,i,j})_{\sigma \in \omega^{n+1}, s \in \omega^{\leq n}, i \neq j}$. For $i < \omega$, let $q_i(x)$ be

$$q(x) \cup \{\varphi(x, b_{\emptyset,i,j}) : j \neq i,\ \mathcal{N} \models \varphi(a_{(i)}, b_{\emptyset,i,j})\} \cup \{\neg\varphi(x, b_{\emptyset,i,j}) : j \neq i,\ \mathcal{N} \models \neg\varphi(a_{(i)}, b_{\emptyset,i,j})\}.$$

Note that each q_i extends q. Moreover, if $i \neq j$ then $q_i \not\sim q_j$ since $\mathcal{N} \models \varphi(a_{(i)}, b_{\emptyset,i,j}) \leftrightarrow \neg\varphi(a_{(j)}, b_{\emptyset,i,j})$. Finally, for a fixed $i < \omega$, if we set $a^i_\sigma = a_{i\sigma}$ and $b^i_{s,j,k} = b_{is,j,k}$, then $(a^i_\sigma, b^i_{s,j,k})_{\sigma\in\omega^n,\sigma\in\omega^{<n},j\neq k}$ realizes $\Gamma(q_i, n)$, and so $R^*(q_i) \geq n$ by induction. So $R^*(q) \geq n + 1$.

For the other direction, suppose $R^*(q) \geq n + 1$. So there is $\mathcal{N} \succeq \mathcal{M}$ and types $q_i(x)$ over $\mathcal{N}$, for $i < \omega$, such that $q \subseteq q_i$, $R^*(q_i) \geq n$, and $q_i \not\sim q_j$ for all $i \neq j$. By induction, $\Gamma(q_i, n)$ is consistent for all $i < \omega$, say realized by $(a^i_\sigma, b^i_{s,j,k})_{\sigma\in\omega^n,\sigma\in\omega^{<n},j\neq k}$ (in some bigger extension of $\mathcal{N}$). Set $a_{i\sigma} = a^i_\sigma$ and $b_{is,j,k} = b^i_{s,j,k}$. Then we have a_σ's for all $\sigma \in \omega^{n+1}$, and $b_{s,j,k}$'s for all $s \in \omega^{\leq n}$ and distinct $j, k \in \omega$ *except for* when $s = \emptyset$. To obtain these elements, we fix distinct $j, k < \omega$, and let $b_{\emptyset,j,k} \in N^y$ witness that $q_j \not\sim q_k$. Then, by construction, $(a_\sigma, b_{s,j,k})_{\sigma\in\omega^{n+1},\sigma\in\omega^{\leq n},j\neq k}$ realizes $\Gamma(q, n + 1)$. This finishes the proof of Step 1.

Step 2: R coincides with R^*.
Proof: First, is clear from the definitions that $R \leq R^*$. For the other direction, we will use the following claim.

Claim: For any $\mathcal{M} \models T$ and any type q over M, there is a complete type $p \in S_x(M)$ such that $R^*(p) = R^*(q)$.

Before proving the claim, let us first use it to prove $R^* \leq R$. Arguing by induction, suppose $R^*(q) \geq n+1$. We want to show $R(q) \geq n+1$. By definition of R^*, there is $\mathcal{N} \succeq \mathcal{M}$ and pairwise $\sim$-inequivalent (partial) types q_i over $\mathcal{N}$, for $i < \omega$, such that $R^*(q_i) \geq n$ and q_i extends q. By the claim, there is a complete type $p_i \in S_x(N)$ extending q_i with $R^*(p_i) = R^*(q_i)$. So $R(p_i) \geq n$ for all i by induction. Since the p_i's are still pairwise $\sim$-inequivalent, and extend q, we have $R(q) \geq n + 1$, as desired.

Now the only thing left to do is prove the claim. Given a type $q(x)$ and a formula $\phi(x)$, we use the notation $q \wedge \varphi$ for $q \cup \{\varphi\}$.

Proof of the Claim: First, we observe that if $\mathcal{M} \models T$ then, for any type q over M, and any $\mathcal{L}_M$-formulas $\psi_1(x)$ and $\psi_2(x)$, we have $R^*(q \wedge (\psi_1 \vee \psi_2)) = \max\{R^*(q \wedge \psi_1), R^*(q \wedge \psi_2)\}$. This follows from the pigeonhole principle just like as in Claim 5 in the proof of Lemma 22.1.

Now fix $\mathcal{M} \models T$ and a type q over M. We construct a complete type $p \in S_x(M)$ such that $R^*(p) = R^*(q)$. The strategy is similar to the proof of Proposition 21.9 (and #7 on Examples Sheet 4). Define π to be the set of all $\mathcal{L}_M$-formulas $\psi(x)$ such that $R^*(q \wedge \neg\psi) < R(^*q)$. We claim that $q \cup \pi$ is consistent. For this, fix finitely many $\psi_1, \ldots, \psi_k \in \pi$. Let ψ be $\psi_1 \wedge \ldots \wedge \psi_k$. We want to show that $q \wedge \psi$ is consistent. To see this, note that

$$\begin{aligned} R^*(q) = R^*(q \wedge (\psi \vee \neg\psi)) = R^*(q \wedge (\psi \vee \neg\psi_1 \vee \ldots \vee \neg\psi_k)) \\ = \max\{R^*(q \wedge \psi), R^*(q \wedge \neg\psi_1), \ldots, R^*(q \wedge \neg\psi_k)\}. \end{aligned}$$

Since each ψ_t is in π, it follows that $R^*(q \wedge \psi) = R^*(q)$. In particular, $q \wedge \psi$ is consistent.

We have now shown that $q \cup \pi$ is consistent, and so we can extend it to a complete type $p \in S_x(M)$. We claim that $R^*(p) = R^*(q)$. Since $q \subseteq p$, we have $R^*(p) \leq R^*(q)$. So suppose $R^*(p) < R^*(q)$. By Step 1, and compactness, there is some $\psi(x) \in p$ such that $R^*(\psi) < R^*(q)$. So $R^*(q \wedge \psi) < R^*(q)$, and thus $\neg\psi \in \pi \subseteq p$, which is a contradiction. Therefore $R^*(p) = R^*(q)$, as desired. This finishes the proof of the claim, and thus the proof of Step 2, and thus the proof of the proposition. □

Discussion. There are many similar ranks used throughout model theory in various important ways. The rank $R(q)$ above is usually referred to as the "local infinity rank", and denoted $R(q, \varphi, \omega)$. The use of the word "local" has to do with the fact that an increase in rank is uniformly controlled by a single formula $\varphi(x, y)$. This is in contrast to other ranks such as *Morley rank* or *Lascar rank*, which are not local in this way.

Chapter 17 of Poizat's *A Course in Model Theory* has more details on ranks (some of which require further notions, such as "forking", in order to define). The local infinity rank is dealt with in Section 17.4. There are some cosmetic differences between Poizat's definition (which is more standard) and the one given above, but the results and remarks in that section explain the equivalence via many of the same properties shown in the above proof.

The most important difference between our definition and Poizat's is that the rank is initially defined to be *ordinal-valued*. This is a general feature of most model-theoretic ranks. Indeed, an abstract template for defining ranks often goes as follows:

(*i*) $R(q) \geq 0$ if and only if q is consistent.

(*ii*) If α is a limit ordinal then $R(q) \geq \alpha$ if and only if $R(q) \geq \beta$ for all $\beta < \alpha$.

(*iii*) $R(q) \geq \alpha + 1$ if and only if **some property of extensions of q of rank at least α**.

Then $R(q)$ is defined to be the supremum of all ordinals α such that $R(q) \geq \alpha$. If this is the set of all ordinals, then $R(q) = \infty$.

For local ranks (where in (*iii*) depends uniformly on a single formula), one often gets similar behavior that if $R(q) \geq \omega$ then $R(q) = \infty$. For example, this happens with the local infinity rank (which part of why our definition is the same as the standard one in Poizat's book). This is in contrast to other non-local ranks such as Morley rank and Lascar rank. For these ranks, the formulation of (*iii*) often still comes down to a properties of formulas, but these formulas might change as the rank increases. Thus in these cases it is possible for types to have rank taking other ordinal values like $\omega, \omega + 1, \omega^\omega, \omega_1$, etc.