Part III - Model Theory
Reducts of Theories

This note clarifies the definition of a reduct of theory, which came up in Corollary 15.4.

Definition. Given an an $\mathcal{L}$-theory T and a sub-language $\mathcal{L}_0 \subseteq \mathcal{L}$, the **reduct of T to $\mathcal{L}$** is the set of $\mathcal{L}_0$-sentences φ such that $T \models \varphi$.

Important Note. While stating Corollary 15.4, I first wrote $\varphi \in T$, and then added $T \models \varphi$ as an "alternate option". This makes no difference when the theory in question is closed under logical consequences (e.g. $\mathrm{Th}(\mathcal{M})$ for some $\mathcal{M}$) and, in practice, it is common to identify a theory with its set of logical consequences. But, in general, the right definition is the one above. Here is an example showing the subtlety.

Example. Let $\mathcal{L}$ contain infinitely many distinct constant symbols c_n for all $n \geq 0$. Define $T = \{c_m \neq c_n : m \neq n\}$. Then T is a complete $\mathcal{L}$-theory, and we can think of T as "the theory of infinite sets with labels for a countable subset". Now let $\mathcal{L}_0 = \emptyset$. Then the reduct of T to $\mathcal{L}_0$ is the theory of infinite sets, since T proves the sentence "there are at least n elements" for all $n \geq 1$. But note that T doesn't literally contain any of these sentences.

An important (but easy) fact is that the reduct of a complete $\mathcal{L}$-theory T to a sub-language $\mathcal{L}_0$ is always a complete $\mathcal{L}_0$-theory.

Finally, recall that the notion of a reduct of a *structure* is much more concrete. Given an $\mathcal{L}$-structure $\mathcal{M}$, and some $\mathcal{L}_0 \subseteq \mathcal{L}$, the reduct of $\mathcal{M}$ to $\mathcal{L}_0$ is the $\mathcal{L}_0$-structure obtained by forgetting the interpretations for anything not in $\mathcal{L}_0$. So one could rephrase Corollary 15.4 as follows.

Corollary. *Assume $\mathcal{L}$ is countable, and suppose $\mathcal{M}$ is an $\mathcal{L}$-structure such that* $\mathrm{Th}(\mathcal{M})$ *is $\aleph_0$-categorical. Let $\mathcal{M}_0$ be the reduct of $\mathcal{M}$ to some $\mathcal{L}_0 \subseteq \mathcal{L}$. Then* $\mathrm{Th}(\mathcal{M}_0)$ *is $\aleph_0$-categorical.*

This is how we applied the corollary in the proof of Vaught's theorem on $I(T, \aleph_0) \neq 2$. (And this will be our only use of the corollary.)