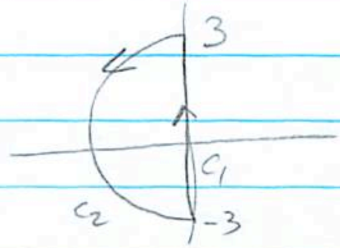
FINAL REVIEW

⓪ Calculate $\oint_D (y\,dx + 2x\,dy)$ where C is the left half circle $x^2+y^2=9$ followed by the segment joining $(0,-3)$ and $(0,3)$ oriented counterclockwise in 2 ways

a) Using Green's thm

b) Calculating directly the line integral.

Sol.



a) $\oint_D P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$

$= \iint_D (2-1)\,dA = \text{Area} = \frac{\pi \cdot 9}{2}$

b) $\oint_D = \int_{C_1} + \int_{C_2}$

On C_1: $\begin{cases} x=0 \\ y=t \end{cases}$ t from -3 to 3

$\frac{dx}{dt}=0$, $\frac{dy}{dt}=1$

$\int_{C_1} = \int_{-3}^{3} (t\cdot 0 + 2\cdot 0\cdot 1)\,dt = 0$

$x = 3\cos t$, $y = 3\sin t$ t from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$

$\int_{C_2} = \int_{\pi/2}^{3\pi/2} 3\sin t\cdot(-3\sin t) + 6\cos t\cdot 3\cos t$

$-9\sin^2 + 18$

$= \int_{\pi/2}^{3\pi/2} -9\sin^2 + 18\cos^2 = \int_{\pi/2}^{3\pi/2} -9\,\frac{1-\cos 2t}{2} + 18\,\frac{1+\cos 2t}{2} = \int_{\pi/2}^{3\pi/2} \frac{9}{2} + \left(\frac{9}{2}+\frac{18}{2}\right)\cos 2t$

$= \frac{9}{2}\pi$ ✓

1. $\underline{F}(x,y) = (2x+y)\,\underline{i} + (x+6y)\,\underline{j}$

a) Let C be an arbitrary path (pw smooth) from (0,0) to (1,1)
Show that $\int_C \underline{F}\cdot d\underline{r} = \int_S F\cdot dr$ where S is the line segm from (0,0) to (1,1)

Sol. $P_y = 1 = Q_x$
Dom - simply conn $\mapsto$ cons $\Rightarrow$ path indep

b) Find the functions f so that $F = \nabla f$

$$\begin{cases} f_x = 2x+y \\ f_y = x+6y \end{cases} \qquad f = x^2+xy+g(y)$$

$$x + g'(y) = x+6y \qquad g(y) = 3y^2 + C$$

$$f(x,y) = x^2+xy+3y^2+C$$

c) Evaluate $\int_C F\cdot dr = f(1,1) - f(0,0) = 3$

d) Evaluate $\int_S F\cdot dr$; S = the circle $x^2+y^2=4$

2. Calculate $\int_C (x+y^2+z)\,ds$ where C is the line segment from (0,1,0) to (1,2,3)

$ds = \sqrt{dx^2+dy^2+dz^2}$

$ds = \frac{ds}{dt}dt = \sqrt{1+1+9}\,dt = \sqrt{10}\,dt$

$\underline{r} = (1-t)\underline{r_0} + t\underline{r_1}$ t from 0 to 1
$= (1-t)\cdot\langle 0,1,0\rangle + t\langle 1,2,3\rangle$
$= \langle 0,1-t,0\rangle + \langle t,2t,3t\rangle$
$= \langle t, 1+t, 3t\rangle$

$$\int_0^1 (t^2+1+t+3t)\sqrt{10}\,dt = \sqrt{10}\int_0^1 (t^2+4t+1)\,dt = \sqrt{10}\left(\frac{1}{3}+2+1\right)$$

(3) Use spherical coord to calculate $\iiint_E x \, dV$

where E is the solid between the spheres of radius 1 and 2, above the cone $z = \sqrt{x^2+y^2}$

$1 \le \rho \le 2$

$z \geq \sqrt{x^2+y^2} \quad \rho\cos\varphi \geq \rho\sin\varphi \quad \cos\varphi \geq \sin\varphi$

$0 \le \varphi \le \frac{\pi}{4}$

$$\int_1^2 \int_0^{\pi/4} \int_0^{2\pi} \rho\sin\varphi\cos\theta \, \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$= \frac{1}{4}\rho^4\Big|_1^2 \cdot 2\pi \cdot \int_0^{\pi/4} \sin^2\varphi\cos\varphi = \frac{1}{4}(2^4-1)\cdot 2\pi \cdot \frac{1}{3}\sin^3\varphi\Big|_0^{\pi/4}$$

(4) Calc. $\iiint_E x \, dV$, E = inside the sphere rad 1, in the octant $x>0, y>0, z<0$

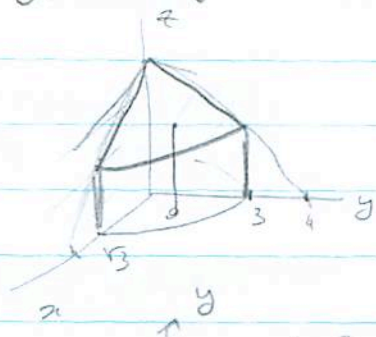
$0 \le \rho \le 1$

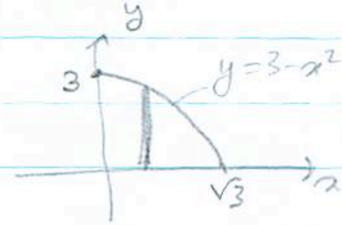
$0 \le \theta \le \pi/2$

$\frac{\pi}{2} \le \varphi \le \pi$

⑤ Use a $\iiint$ to calc the vol of the solid bd by the cylinder $y=3-x^2$ and the planes

$x=0$, $y=0$, $z=0$, $2x+y+z=4$





$$V=\iiint_E dV=\iint\left(\int_0^{4-2x-y} dz\right) dA$$

$$=\int_0^{\sqrt{3}}\int_0^{3-x^2}\int_0^{4-2x-y} dz\, dy\, dx$$

$$=\int_0^{\sqrt{3}}\int_0^{3-x^2}(4-2x-y)\, dy\, dx$$

$$=\int_0^{\sqrt{3}} 4y-2xy-\tfrac{1}{2}y^2\Big|_{y=0}^{y=3-x^2} dx$$

$$=\int_0^{\sqrt{3}} 4(3-x^2)-2x(3-x^2)-\tfrac{1}{2}(3-x^2)^2\, dx=\ldots$$

⑥ $f(x,y) = x^2 + y^2 - 4y$

a) Find the critical pts and use the 2^{nd} deriv test

$f_x = 2x = 0$ $\quad (0,2)$

$f_y = 2y - 4 = 0$

$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 0 \quad \Delta = 4 > 0,\ f_{xx} > 0$ min

b) Use the lagr mult to find the extrema on the ellipse $x^2 + 2y^2 = 9$

$\begin{cases} 2x = 2\lambda x \\ \cancel{4}y - 4 = 2\lambda y \\ x^2 + y^2 = 9 \end{cases}$ $\quad \begin{cases} x(\lambda - 1) = 0 \\ \cancel{4}y - 4 = 2\lambda y \\ x^2 + y^2 = 9 \end{cases}$

(I) $x = 0 \Rightarrow y = \pm 3$

(II) $x \neq 0 \Rightarrow \lambda = 1 \Rightarrow \cancel{4}y - 4 = 2y \quad 2y = 4$

$y = 2$

$x^2 + 2 \cdot 4 = 9$

$x = \pm 1 \quad (\pm 1, 2)$

$f(0,\sqrt{3}) = 3 - 4\sqrt{3}$

$f(0,-\sqrt{3}) = 3 + 4\sqrt{3}$ Max

$f(1,2) = 1 + 4 - 8 = -3 = f(-1,2)$

$3 - 4\sqrt{3} \approx 3 - 4 \cdot 1.71 = 3 - 6,84 = -3.84$ min

c) Find the extrema of f on the elliptical disk $x^2 + 2y^2 \leq 9$ using a, b. Explain how these results are useful.

a) what is the domain of

⑦ b) Write the linear approx of $f(x,y,z) = x \ln(2x+3y+4z)$
at the point $(1,1,-1)$

$f(1,1,-1) = 1 \ln(2+3-4) = 1 \ln 1 = 0$

⑧ Show that $u(x,y) = f(x-2y)$ with f twice diff
with f'' cont

satisfies $4u_{xx} - u_{yy} = 0$

$u_x = f'(x-2y)$

$u_{xx} = f''(x-2y)$

$u_y = -2f'(x-2y)$

$u_{yy} = 4f''(x-2y)$