2.12. **Eigenspaces.** Consider an $n \times n$ matrix M with entries in F, with eigenvalues $\lambda_1, \ldots, \lambda_n$ in F.

Definition 11. *The set*

$$V_{\lambda_j} = \{\mathbf{x} \in F^n \,|\, M\mathbf{x} = \lambda_j \mathbf{x}\}$$

is called **the eigenspace** *of M associated to the eigenvalue λ_j.*

Exercise. Show that V_{λ_j} is the null space of the transformation $M - \lambda I$ and that V_{λ_j} is a subspace of F^n.

Note that all the *nonzero* vectors in V_{λ_j} are eigenvectors of M corresponding to the eigenvalues λ_j.

Definition 12. *A subspace V is called an* **invariant subspace** *for M if $M(V) \subset V$ (which means that if $\mathbf{x} \in V$ then $M\mathbf{x} \in V$).*

The following Remark gathers main features of eigenspaces; their proof is left to the reader.

Remark. 1. Each V_{λ_j} is an invariant subspace for M.

2. $V_{\lambda_j} \cap V_{\lambda_l} = \{\mathbf{0}\}$ if $\lambda_j \neq \lambda_l$.

3. Denote by $\lambda_1, \ldots, \lambda_k$ the distinct eigenvalues of M and by r_j the multiplicity of the eigenvalue λ_j, for each $j = 1, \ldots, k$; it is clear that

$$\det(M - \lambda I) = \prod_{j=1}^{k} (\lambda_j - \lambda)^{r_j} \text{ and } r_1 + \ldots + r_k = n$$

Then

$$\dim V_{\lambda_j} \leq r_j$$

4. M is diagonalizable in F^n if and only if $\dim V_{\lambda_j} = r_j$ for all $j = 1, \ldots, k$ and then

$$V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_k} = F^n$$

Example. Consider the matrix

$$M := \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} \tag{19}$$

Its characteristic polynomials is

$$\det(M - \lambda I) = -\lambda^3 + 3\,\lambda^2 - 4 = -\,(\lambda + 1)\,(\lambda - 2)^2$$

so $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = 2$ is a double eigenvalue. The eigenspace V_{λ_1} is one dimensional, spanned by an eigenvector, which, after a simple calculation turns out to be $\mathbf{v}_1 = (0, 1, 1)^T$. If the eigenspace V_{λ_2} is two-dimensional (which is not guaranteed) then the matrix M is diagonalizable. A simple

calculation shows that there are two independent eigenvectors corresponding to the eigenvalue $\lambda_2 = 2$, for example $\mathbf{v}_2 = (1,0,1)^T$ and $\mathbf{v}_3 = (2,1,0)^T$ (the null space of $M - \lambda_2 I$ is two-dimensional). Let

$$S = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

then

$$S^{-1}MS = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2.13. **Real matrices with complex eigenvalues; decomplexification.**

2.13.1. *Complex eigenvalues of real matrices.* For an $n \times n$ matrix with real entries, if we want to have n guaranteed eigenvalues, then we have to accept working in $\mathbb{C}^n$. Otherwise, if we want to restrict ourselves to working only with real vectors, then we have to accept that we may have fewer (real) eigenvalues, or perhaps none.

Complex eigenvalues of real matrices come in pairs: if λ is an eigenvalue of M, then so is its complex conjugate $\overline{\lambda}$ (since the characteristic equation has real coefficients). Also, if $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue λ, then $\overline{\mathbf{v}}$ is eigenvector corresponding to the eigenvalue $\overline{\lambda}$ (check!). The real and imaginary parts of $\mathbf{v}$ span a plane where the linear transformation acts by rotation, and a possible dilation. Simple examples are shown below.

Example 1: rotation in the xy-plane. Consider a rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{20}$$

To find its eigenvalues calculate

$$\det(R_\theta - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = \lambda^2 - 2\cos\theta + 1$$

hence the solutions of the characteristic equations $\det(R_\theta - \lambda I) = 0$ are $\lambda_{1,2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$. It is easy to see that $\mathbf{v}_1 = (i,1)^T$ is the eigenvector corresponding to $\lambda_1 = e^{i\theta}$ and $\mathbf{v}_2 = (-i,1)^T$ is the eigenvector corresponding to $\lambda_2 = e^{-i\theta}$.

Example 2: complex eigenvalues in $\mathbb{R}^3$. Consider the matrix

$$M = \begin{bmatrix} 1 - \frac{1}{2}\sqrt{3} & -\frac{5}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & 1 + \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

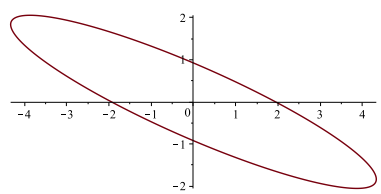
Its characteristic polynomial is

$$\det(M - \lambda I) = -\lambda^3 - 2\,\lambda^2 + 4\,\lambda - 16 = -\,(\lambda + 4)\left(\lambda^2 - 2\,\lambda + 4\right)$$

and its eigenvalues are: $\lambda_{1,2} = 1 \pm i\sqrt{3} = 2e^{\pm i\pi/3}$ and $\lambda_3 = -4$, and corresponding eigenvectors $\mathbf{v}_{1,2} = (-1 \pm 2i, 1, 0)^T$, and $\mathbf{v}_3 = \mathbf{e}_3$. The matrix $S = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ diagonalized the matrix: $S^{-1}MS$ is the diagonal matrix, having the eigenvalues on the diagonal, but all these are complex matrices.

To understand how the matrix acts on $\mathbb{R}^3$, we consider the real and imaginary parts of $\mathbf{v}_1$: let $\mathbf{x}_1 = \Re\mathbf{v}_1 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2) = (-1, 1, 0)^T$ and $\mathbf{y}_1 = \Im\mathbf{v}_1 = \frac{1}{2i}(\mathbf{v}_1 - \mathbf{v}_2) = (2, 0, 0)^T$. Since the eigenspaces are invariant under M, then so is $Sp(\mathbf{x}_1, \mathbf{y}_1)$, over the complex and even over the real numbers (since M has real elements). The span over the real numbers is the xy-plane, and it is invariant under M. The figure shows the image of the unit circle in the xy-plane under the matrix M: it is an ellipse.

FIGURE 1. The image of the unit circle in the xy-plane.



Along the direction of the third eigenvector (the z-axis) the matrix multiples any $c\,\mathbf{e}_3$ by -4.

In the basis $\mathbf{x}_1, \mathbf{y}_1, \mathbf{v}_3$ the matrix of the linear transformation has its simplest form: using $S_{\mathbb{R}} = [\mathbf{x}_1, \mathbf{y}_1, \mathbf{v}_3]$ we obtain the matrix of the transformation in this new basis as

$$S_{\mathbb{R}}^{-1} M S_{\mathbb{R}} = \begin{bmatrix} 1 & \sqrt{3} & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

and the upper 2×2 block represents the rotation and dilation $2R_{-\pi/3}$.

2.13.2. *Decomplexification.* Suppose the $n \times n$ matrix M has real elements, eigenvalues $\lambda_1, \ldots, \lambda_n$ and n independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then M is diagonalizable: if $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ then $S^{-1}MS = \Lambda$ where Λ is a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on its diagonal.