1. Check that $\mathbb{R}^2$ with the usual, component-wise, addition and scalar multiplication is a vector space over the scalars $\mathbb{R}$.

2. a) Show that the set of continuous functions over the interval $[a, b]$
   \[ C[a, b] = \{ f : [a, b] \rightarrow \mathbb{R} \mid f \text{ continuous on } [a, b] \} \]
   is a linear space over the scalars $\mathbb{R}$.
   
   b) Show that $C_0[a, b] := \{ f \in C[a, b] \mid f(a) = f(b) = 0 \}$ is a subspace of $C[a, b]$.
   
   c) Is $\{ f \in C[a, b] \mid \int_a^b f(x) \, dx = 0 \}$ a subspace of $C[a, b]$? Justify.

3. a) Let $X$ and $Y$ be two vector spaces over $F$. Let
   \[ X \oplus Y = \{(x, y) \mid x \in X, y \in Y\} \]
   Show that $X \oplus Y$ is a vector space over $F$ with addition and scalar multiplication defined component-wise, as
   \[(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad c(x, y) = (cx, cy) \]
   $X \oplus Y$ is called the (external) direct sum of the vector spaces $X$ and $Y$.
   
   b) Given $\{v_i\}_{i=1,...,n}$ a basis for $X$, and $\{w_j\}_{j=1,...,k}$ a basis for $Y$, find a basis for $X \oplus Y$. Express the dimension of $X \oplus Y$ in terms of $\dim X$ and $\dim Y$.

4. In each of the following cases, establish whether or not the given set of vectors is linearly independent or linearly dependent in the given vector/linear space. Explain.
   
a) 1, cos $t$, cos $2t$, ..., cos $nt$ in $C[0, 2\pi]$.
   
b) $p(t) = (t-1)(t-2)(t-3)$, $q(t) = t(t-2)(t-3)$, $r(t) = t(t-1)(t-3)$, $s(t) = t(t-1)(t-2)$ in $\mathcal{P}$.
   
c) $t^{\sqrt{2}}$, $t^e$, $t^\pi$ in $C(0, \infty)$
   
d) cosh $x$, cosh($x-1$) in $C(\mathbb{R})$.
   
e) (1, 1, 1, 1), (0, 2, 1, −1), (2, −4, −1, 5) in $\mathbb{R}^4$.
   
f) (1, 1, 1, 1), (0, 2, 1, −1), (2, −1, 1, −1) in $\mathbb{R}^4$. 
