Justify all your answers!

1. Let $A_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, $Y = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $Z = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

a1) True or False? $Y \in \text{Sp}(A_1, A_2, A_3)$.

a2) True or False? $Z \in \text{Sp}(A_1, A_2, A_3)$.

b) Find a basis for $\text{Sp}(A_1, A_2, A_3)$.

2. Consider the following polynomials in $\mathcal{P}_3$ (the linear space of polynomials of degree at most three) $\phi_1(t) = t^3$, $\phi_2(t) = t^2(1 - t)$, $\phi_3(t) = t(1 - t)^2$, $\phi_4(t) = (1 - t)^3$.

Show that every polynomial $p \in \mathcal{P}_3$ has a unique representation $p(t) = \sum_{j=1}^{4} c_j \phi_j(t)$ with $c_1, \ldots, c_4$ constants.

3. Let $\mathcal{B}_N$ = the set of all linear combinations of $e^{ikt}$, $k = -N, \ldots, 0, \ldots, N$, with complex coefficients. (Such linear combinations form a set of “band limited” functions.)

It is easy to see that $\mathcal{B}_N$ is a linear space of functions over $\mathbb{C}$, and it can be shown that $e^{ikt}$, $k = -N, \ldots, 0, \ldots, N$ are linearly independent. Assume these are true.

a) What is $\dim \mathcal{B}_N$?

b) Show that $\{1, \cos t, \ldots, \cos Nt, \sin t, \ldots, \sin Nt\}$ is a basis for $\mathcal{B}_N$.

4. Consider the transformation of $\mathbb{R}^3$ given by the matrix multiplication $x \rightarrow Mx$ where

$$M = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

a) Find the range and the null space of this transformation, describing these subspaces by giving a basis and as geometrical objects in $\mathbb{R}^3$.

b) Find all the vectors $b \in \mathbb{R}^3$ for which the system $Mx = b$ is soluble.

c) Find the general solution to $Mx = 0$.

(More problems on next page)
5. Let $\mathcal{M}_{2,2}(\mathbb{R})$ be the linear space of all $2 \times 2$ matrices with the real entries.
   a) Prove that $\mathcal{M}_{2,2}(\mathbb{R})$ has dimension 4 and find a basis.
   b) Show that the trace
   
   $$Tr : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}, \quad Tr(A) = A_{11} + A_{22}$$
   
   is a linear functional and use this to show that the set of matrices of zero trace form a subspace in $\mathcal{M}_{2,2}(\mathbb{R})$.
   c) What is the dimension of the subspace of matrices of zero trace?

6. Let $V$ and $W$ both be subspaces of a given vector space.
   True or False? Their intersection $V \cap W = \{x : x \in V \text{ and } x \in W\}$ is also a subspace.
   (To justify, you need to prove it if True, or find a counterexample if False.)