1. a) For the basis dual to the standard basis \( \{e_1, e_2, e_3\} \) give explicit formulas for \( e'_1(x_1, x_2, x_3) \), \( e'_2(x_1, x_2, x_3) \), \( e'_3(x_1, x_2, x_3) \).

b) Write the linear functional \( f : \mathbb{R}^3 \to \mathbb{R} \) defined by

\[
f(x_1, x_2, x_3) = a_1 x_1 + a_2 x_2 + a_3 x_3
\]

as a linear combination of \( e'_1, e'_2, e'_3 \).

2. Consider the basis for \( \mathbb{R}^3 \): \( B = \{v_1, v_2, v_3\} \) where

\[
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
2 \\
2 \\
0
\end{bmatrix}
\]

a) Find the dual basis, giving formulas in the form (1).

b) Find the row vector representation of the dual basis in the standard basis of \( \mathbb{R}^3 \).

*Hint:* you can do calculations based on definitions, or you can use the result of Section 1.4.1 of the notes on The Dual Space.

3. Numerical quadrature. Numerical integration of functions uses only the values of functions at certain sample points. In the space \( \mathcal{P}_n \) of polynomials of degree at most \( n \) there is an exact numerical quadrature: show that if \( s_0, s_1, \ldots, s_n \) are some sample points in an interval \([a, b]\) then there are some numbers \( c_0, c_1, \ldots, c_n \) so that

\[
\int_a^b p(x) \, dx = c_0 p(s_0) + a_1 p(s_1) + \ldots + c_n p(s_n) \quad \text{for all polynomials } p \in \mathcal{P}_n
\]

Specify a way to calculate the numbers \( c_0, c_1, \ldots, c_n \), but do not calculate them.

(continued on next page)
4. Consider the function \( L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}^2 \) given by the formula
\[
L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + 2d \\ b - c \end{bmatrix}
\]
a) Show that \( L \) is a linear transformation.
b) Give formulas for all the matrices in \( \mathcal{N}(L) \).
c) Find a basis for \( \mathcal{N}(L) \).
d) Determine the nullity and the rank of \( L \).

5. a) Show that \( 1, x-a, (x-a)^2 \) is a basis for \( \mathcal{P}_2 \) and that \( E_a, \frac{d}{dx} \bigg|_{x=a}, \frac{1}{2} \frac{d^2}{dx^2} \bigg|_{x=a} \) is the dual basis; the notation is the usual one:
\[
(E_a, p) = p(a), \quad \left(\frac{d}{dx} \bigg|_{x=a}, p\right) = p'(a), \quad \left(\frac{1}{2} \frac{d^2}{dx^2} \bigg|_{x=a}, p\right) = \frac{1}{2}p''(a)
\]
b) Let \( T : \mathcal{P}_2 \to \mathcal{P}_4 \) be a linear transformation such that
\[T(1) = x^4, \ T(x + 1) = x^3 - 2x, \ T((x + 1)^2) = x\]
Find \( T(p) \) and \( T(q) \) where \( p = x^2 + 5x - 1 \) and \( q = a_0 + a_1x + a_2x^2 \).

6. Let \( T : \mathcal{P}_4 \to \mathcal{P}_3 \) be given by
\[
(2) \quad T(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) = (a_0 - a_1 + 2a_2 - a_3 + a_4)
+ (-a_0 + 3a_1 - 2a_2 + 3a_3 - a_4)x
+ (2a_0 - 3a_1 + 5a_2 - a_3 + a_4)x^2
+ (3a_0 - a_1 + 7a_2 + 2a_3 + 2a_4)x^3
\]
a) Find a basis for \( \mathcal{R}(T) \).
b) Show that \( T \) is not one-to-one.