2.6. **Projections.** The prototype of an oblique projection is the shadow cast by objects on the ground (when the sun is not directly vertical). Mathematically, we have a plane (the ground), a direction (direction of the rays of light, which are approximately parallel, since the sun is very far), and the projection of a person along the light rays is its shadow on the ground.

More generally, we can have projections parallel to higher dimensional subspaces (they are quite useful in image processing among other applications).

The simplest way to define a projection mathematically is

**Definition 24.** Given a linear space $V$, a projection is a linear transformation $P : V \rightarrow V$ so that $P^2 = P$.

**Exercise.** Show that the eigenvalues of a projection can be only 0 or 1.

The subspace $R = \mathcal{R}(P)$ is the subspace on which $P$ projects, and the projection is parallel to (or, along) the subspace $N = \mathcal{N}(P)$.

If $y \in R$ then $Py = y$ since, for any $y \in R$, then $y = Px$, hence $Py = P(Px) = P^2x = Px = y$.

Since $\dim \mathcal{R}(P) + \dim \mathcal{N}(P) = n$ it follows that $R \oplus N = V$. Therefore any $x \in V$ has the form $x = x_R + x_N$ where $x_R \in R$ and $x_N \in N$ and $Px = x_R$. This is a true projection!

### 2.6.1. Geometrical characterization of projections.

Conversely, given two subspaces $R$ and $N$ so that $R \oplus N = V$, a projection onto $R$, parallel to $N$ is defined geometrically as follows.

**Proposition 25.** If $R \oplus N = V$, then for any $x \in V$, there is a unique vector $Px \in R$ so that $x - Px \in N$.

$P$ is a projection (the projection onto $R$ along $N$).

The proof is immediate, since any $x \in V$ can be uniquely written as $x = x_R + x_N$ with $x_R \in R$ and $x_N \in N$, hence $Px = x_R$. □

Of course, the projection is orthogonal if $N = R^\perp$.

**Example.** In $V = \mathbb{R}^3$ consider the projection on $R = \text{Sp}(e_1, e_2)$ (the $x_1x_2$-plane) along a line $N = \text{Sp}(n)$ where $n$ is a vector not in $R$, $n = (n_1, n_2, n_3)$ with $n_3 \neq 0$. We construct the projection geometrically. Given a point $x = (x_1, x_2, x_3)$, the line parallel to $n$ and passing through $x$ is $x + tn$, $t \in \mathbb{R}$. The line intersects the $x_1x_2$-plane for $t = -\frac{x_3}{n_3}$. Hence $Px = x - \frac{x_3}{n_3}n$, or

$$Px = x - \frac{(x, u)}{(n, u)}n,$$

where $u \perp R$. 

2.6.2. The matrix of a projection. Let \( \{u_1, \ldots, u_r\} \) be a basis for \( R \), and \( \{u_{r+1}, \ldots, u_n\} \) a basis for \( S \); then \( \{u_1, \ldots, u_n\} \) is a basis for \( V \), and in this basis the matrix of the sought-for projection is

\[
M_P = \begin{bmatrix}
I & O \\
O & O
\end{bmatrix}
\]

To find the matrix in another basis, say, for example, \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \), and the vectors \( u_j \) are given by their coordinates in the standard basis, then the transition matrix (from old basis to standard basis) is \( S = [u_1, \ldots, u_n] \) and then the matrix of \( P \) in the standard basis is \( S \cdot M_P \cdot S^{-1} \).

An alternative formula is\(^2\)

**Proposition 26.** Let \( R \) and \( N \) be subspaces of \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \) such that \( R \oplus N = V \).

Let \( \{u_1, \ldots, u_r\} \) be a basis for \( R \) and \( \{v_1, \ldots, v_r\} \) be a basis for \( N \). Let \( A = [u_1, \ldots, u_r] \) and \( B = [v_1, \ldots, v_r] \) (where the vectors are expressed as their coordinates in the standard basis).

Then the matrix (in the standard basis) of the projection on \( R \) parallel to \( N \) is

\[
A(B^*A)^{-1}B^*
\]

**Proof.**

Let \( x \in V \). We search for \( Px \in R \), therefore, for some scalars \( y_1, \ldots, y_r \)

\[
Px = \sum_{j=1}^{r} y_j u_j = [u_1, \ldots, u_r] \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix} = Ay
\]

Since \( x - Px \in N = (N^\perp)^\perp \) this means that \( x - Px \perp N^\perp \), therefore

\[
\langle v_k, x - Px \rangle = 0 \text{ for all } k = 1, \ldots, r.
\]

Hence

\[
0 = \langle v_k, x - Px \rangle = \langle v_k, x \rangle - \langle v_k, Px \rangle = \langle v_k, x \rangle - \langle v_k, \sum_{j=1}^{r} y_j u_j \rangle
\]

so

\[
0 = \langle v_k, x \rangle - \sum_{j=1}^{r} y_j \langle v_k, u_j \rangle, \quad k = 1, \ldots, r
\]

(17)

Note that element \((j, k)\) of the matrix \( B^*A \) is precisely \( \langle v_k, u_j \rangle = v_k^* u_j = v_k^T u_j \) hence the system (17) is

\[
B^*Ay = B^*x
\]

which we need to solve for \( y \).

**Claim:** The \( r \times r \) matrix \( B^*A \) is invertible.

Its justification will have to wait for a little while. Assuming for the moment the claim is true, then it follows that $\mathbf{y} = (B^*A)^{-1}B^*\mathbf{x}$. Therefore $P\mathbf{x} = PA\mathbf{y} = A(B^*A)^{-1}B^*\mathbf{x}$. $\square$

2.7. More about orthogonal projections. We have defined projections in §2.6 as endomorphisms $P : V \to V$ satisfying $P^2 = P$ and we saw that in the particular case when $\mathcal{N}(P) = \mathcal{R}(P)^\perp$ we have orthogonal projections, characterized in Theorem 18.

The first theorem shows that a projection is orthogonal if and only if it is self-adjoint:

**Theorem 27.** Let $V$ be a finite dimensional inner product space. A linear transformation $P : V \to V$ is an orthogonal projection if and only if $P^2 = P$ and $P = P^*$.  

**Proof.**  

We already showed that $P^2 = P$ means that $P$ is a projection. We only need to check that $P = P^*$ if and only if $P$ is an orthogonal projection, that is, if and only if $\mathcal{N}(P) = \mathcal{R}(P)^\perp$.

Assume $P = P^*$. Let $\mathbf{x} \in \mathcal{N}(P)$ and $P\mathbf{y} \in \mathcal{R}(P)$. Then $\langle \mathbf{x}, P\mathbf{y} \rangle = \langle P^*\mathbf{x}, \mathbf{y} \rangle = \langle P\mathbf{x}, \mathbf{y} \rangle = 0$ therefore $\mathcal{N}(P) \subset \mathcal{R}(P)^\perp$ and since $\mathcal{N}(P) \oplus \mathcal{R}(P) = V$ then $\mathcal{N}(P) = \mathcal{R}(P)^\perp$.

The converse implication is similar. $\square$

2.8.1. Fundamental subspaces of linear transformations.

**Theorem 28.** Let $L : U \to V$ be a linear transformation between two inner product spaces $(V, \langle \cdot, \cdot \rangle_V), (U, \langle \cdot, \cdot \rangle_U)$. Then

(i) $\mathcal{N}(L^*) = \mathcal{R}(L)^\perp$

(ii) $\mathcal{R}(L^*)^\perp = \mathcal{N}(L)$

*Note:* The theorem is true in finite or infinite dimensions. Of course, in finite dimensions also: $\mathcal{N}(L^*) = \mathcal{R}(L)$ and $\mathcal{R}(L^*) = \mathcal{N}(L)^\perp$ (but in infinite dimensions extra care is needed).

**Remark 29.** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. We have:

$\langle x, v \rangle = 0$ for all $v \in V$ if and only if $x = 0$.

This is easy to see, since if we take in particular $v = x$ then $\langle x, x \rangle = 0$ which implies $x = 0$.

*Proof of Theorem 28.*

(i) We have $y \in \mathcal{R}(L)^\perp$ if and only if $\langle y, Lx \rangle = 0$ for all $x \in U$ if and only if $\langle L^*y, x \rangle = 0$ for all $x \in U$ if and only if (by Remark 29) $L^*y = 0$ hence $y \in \mathcal{N}(L^*)$.

(ii) follows from (i) after replacing $L$ by $L^*$ and using (16). \(\square\)

**Exercise.** Show that if $P$ is a projection onto $R$ along $N$ then $P^*$ is a projection onto $N^\perp$ along $R^\perp$.

2.8.2. The four fundamental subspaces of a matrix. Recall that if $M$ is the matrix associated $L$ in two bases which are orthonormal, then the matrix associated to $L^*$ (in the same bases) is the conjugate transpose of $M$, denoted $M^* = \overline{M^T} = \overline{M^T}$.

To transcribe Theorem 28 in language of matrices, once orthonormal bases are chosen $L$ becomes matrix multiplication taking (coordinate) vectors from $F^n$ to (coordinate) vectors $Mx \in F^m$, where $M$ is an $m \times n$ matrix with elements in $F$ and the inner products on the coordinate spaces are the dot products if $F = \mathbb{R}$ and conjugate-dot products in $F = \mathbb{C}$.

Recall that $\mathcal{R}(M)$ is the column space of $M$, and $\mathcal{N}(M)$ is the right null space of $M$: the space of all $x$ so that $Mx = 0$.

Recall that the left null space of $M$ is defined as the space of all vectors so that $y^T M = 0$.

*In the real case,* when $F = \mathbb{R}$, $M^* = M^T$. Then $y$ belongs to the left null space of $M$ means that $y \in \mathcal{N}(M^T)$. Also, the row space of $M$ is $\mathcal{R}(M^T)$.

Theorem 28 states that:

**Theorem 30.** Let $M$ be an $m \times n$ real matrix. Then: