3.7. **Solving linear systems by minimization.** A scalar equation $Ax = b$ can be solved by minimization: its solution coincides with the point $x = x_m$ where the parabola $p(x) = \frac{1}{2}Ax^2 - bx$ has a minimum (assuming $A > 0$). To generalize this minimization solution to higher dimensions, note that the derivative of p is $p'(x) = Ax - b$, whose critical point is the solution of the linear equation $Ax = b$, and the second derivative is $p''(x) = A > 0$, which ensures that the critical point is a minimum. The idea is then to construct a function $p(\mathbf{x})$ whose gradient is $A\mathbf{x} - \mathbf{b}$, and Hessian is A:

Theorem 31. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a positive definite matrix, and $\mathbf{b} \in \mathbb{R}^n$.*

The quadratic form

$$p(\mathbf{x}) = \frac{1}{2}\langle \mathbf{x}, A\mathbf{x}\rangle - \langle \mathbf{x}, \mathbf{b}\rangle$$

has a global minimum.

The minimum is attained at a point $\mathbf{x}_m$ satisfying $A\mathbf{x}_m = \mathbf{b}$ and the minimum value is $p(\mathbf{x}_m) = -\frac{1}{2}\langle A^{-1}\mathbf{b}, \mathbf{b}\rangle$.

Proof:

The gradient of $p(\mathbf{x})$ equals $A\mathbf{x} - \mathbf{b}$ since from

$$p(\mathbf{x}) = p(x_1, \dots, x_n) = \frac{1}{2}\sum_{i,j=1,\dots,n} A_{ij}x_ix_j - \sum_{i=1,\dots,n} b_ix_i$$

we get, using that A is symmetric,

$$\frac{\partial p}{\partial x_k} = \frac{1}{2}\sum_{j=1,\dots,n} A_{kj}x_j + \frac{1}{2}\sum_{i=1,\dots,n} A_{ik}x_i - b_k = \sum_{j=1,\dots,n} A_{kj}x_j + b_k = (A\mathbf{x})_k - b_k$$

and the Hessian of $p(\mathbf{x})$ is A since $\frac{\partial^2 p}{\partial x_l \partial x_k} = A_{lk}$.

[Here is a more compact way to do this calculation:

$$\frac{\partial p}{\partial x_k} = \frac{1}{2}\langle \frac{\partial}{\partial x_k}\mathbf{x}, A\mathbf{x}\rangle + \frac{1}{2}\langle \mathbf{x}, A\frac{\partial}{\partial x_k}\mathbf{x}\rangle - \langle \frac{\partial}{\partial x_k}\mathbf{x}, \mathbf{b}\rangle$$

$$= \frac{1}{2}\langle \mathbf{e}_k, A\mathbf{x}\rangle + \frac{1}{2}\langle \mathbf{x}, A\mathbf{e}_k\rangle - \langle \mathbf{e}_k, \mathbf{b}\rangle = \langle \mathbf{e}_k, A\mathbf{x}\rangle - b_k = (A\mathbf{x})_k - b_k$$

and the Hessian of $p(\mathbf{x})$ is A since $\frac{\partial^2 p}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j}\langle \mathbf{e}_k, A\mathbf{x}\rangle = \langle \mathbf{e}_k, A\frac{\partial}{\partial x_j}\mathbf{x}\rangle = \langle \mathbf{e}_k, A\mathbf{e}_j\rangle = A_{jk}$.]

Note that the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = \mathbf{x}_m = A^{-1}\mathbf{b}$ (since A has no zero eigenvalues).

Write the Taylor expansion of p at $\mathbf{x} = \mathbf{x}_m$, which stops at the second order terms, since p is a polynomial of degree 2, hence derivatives of higher order vanish:

$$p(\mathbf{x}) = p(\mathbf{x}_m) + \langle \nabla p(\mathbf{x}_m), (\mathbf{x} - \mathbf{x}_m)\rangle + \frac{1}{2}\langle (\mathbf{x} - \mathbf{x}_m), (Hp)(\mathbf{x}_m)\,(\mathbf{x} - \mathbf{x}_m)\rangle$$

$$= p(\mathbf{x}_m) + \langle (A\mathbf{x}_m - \mathbf{b}), (\mathbf{x} - \mathbf{x}_m)\rangle + \frac{1}{2}\langle (\mathbf{x} - \mathbf{x}_m), A\,(\mathbf{x} - \mathbf{x}_m)\rangle$$

$$= p(\mathbf{x}_m) + \frac{1}{2}\langle(\mathbf{x}-\mathbf{x}_m), A\,(\mathbf{x}-\mathbf{x}_m)\rangle \quad > \; p(\mathbf{x}_m) \text{ for all } \mathbf{x} \neq \mathbf{x}_m$$

where the last inequality holds since A is positive definite; this means that $p(\mathbf{x}_m)$ is an absolute minimum. Furthermore

$$p(\mathbf{x}_{min}) = \frac{1}{2}\langle \mathbf{x}_{min}, A\mathbf{x}_{min}\rangle - \langle \mathbf{x}_{min}, \mathbf{b}\rangle = -\frac{1}{2}\langle \mathbf{x}_{min}, \mathbf{b}\rangle = -\frac{1}{2}\langle A^{-1}\mathbf{b}, \mathbf{b}\rangle$$

□

3.8. **Generalized eigenvalue problems.** The following type of problem appears in applications (for example they arise in discretizations of continuous problems):

Problem: given two $n \times n$ matrices A and B find the numbers λ so that

$$A\mathbf{v} = \lambda B\mathbf{v} \quad \text{for some } \mathbf{v} \neq \mathbf{0} \tag{18}$$

Clearly, such numbers λ must satisfy the equation $\det(A-\lambda B) = 0$, which is a polynomial in λ of degree at most n.

Since $\det(A - \lambda B) = \lambda^n \det(\frac{1}{\lambda}A - B)$ wee that the coefficient of λ^n is $\det(-B)$. Therefore, if B is not invertible, the degree of $\det(A - \lambda B)$ is less than n.

On the other hand, if B is invertible, then (18) is equivalent to $B^{-1}A\mathbf{v} = \lambda\mathbf{v}$ so λ and $\mathbf{v}$ are the eigenvalues and eigenvectors of the matrix $B^{-1}A$.

For real matrices, A symmetric and B positive definite, the eigenvalues/vectors of the generalized problem have special properties, which are derived below.

If B is positive definite and real, then $B = M^T M$ for a positive definite real matrix M, by Theorem 23 (iv). Equation (18) is $A\mathbf{v} = \lambda M^T M\mathbf{v}$ therefore $(M^T)^{-1}A\mathbf{v} = \lambda M\mathbf{v}$, where denoting $M\mathbf{v} = \mathbf{y}$ and $C = M^{-1}$ the equation becomes

$$(C^T AC)\mathbf{y} = \lambda\mathbf{y} \tag{19}$$

which is an eigenvalue problem for the symmetric matrix $C^T AC$: there are n real eigenvalues, and n orthonormal eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$. Going back through the substitution, let $\mathbf{v}_j$'s be so that $\mathbf{u}_j = M\mathbf{v}_j$, we find the eigenvectors of the generalized problem (18) as $\mathbf{v}_1, \dots, \mathbf{v}_n$ where

$$\delta_{ij} = \langle \mathbf{u}_j, \mathbf{u}_i\rangle = \langle M\mathbf{v}_j, M\mathbf{v}_i\rangle = \langle \mathbf{v}_j, M^T M\mathbf{v}_i\rangle = \langle \mathbf{v}_j, B\mathbf{v}_i\rangle$$

and therefore

$$\langle \mathbf{v}_j, B\mathbf{v}_i\rangle = \delta_{ij}, \quad \text{for } i, j = 1, \dots, n \tag{20}$$

meaning that the $\mathbf{v}_j$ are B-orthonormal, i.e. orthonormal with respect to the inner product $\langle\cdot,\cdot\rangle_B$, see Theorem 23 (iii).

Also

$$\langle \mathbf{v}_j, A\mathbf{v}_i\rangle = \langle \mathbf{v}_j, \lambda_i B\mathbf{v}_i\rangle = \lambda_i\langle \mathbf{v}_j, M^T M\mathbf{v}_i\rangle = \lambda_i\langle \mathbf{v}_j, B\mathbf{v}_i\rangle = \lambda_i\ \delta_{ij}$$

so

$$\langle \mathbf{v}_j, A\mathbf{v}_i \rangle = \delta_{ij}\lambda_i, \quad \text{for } i, j = 1, \ldots, n \tag{21}$$

meaning that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are also A-orthogonal.

In matrix notation, if $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is the matrix with columns the generalized eigenvectors, then (21) is $S^T AS = \Lambda$, and (20) is $S^T BS = I$.

The matrices A and B are simultaneously diagonalized by a congruence transformation! We proved:

Theorem 32. *Consider the real matrices: A symmetric and B positive definite. Then the quadratic forms*

$$q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle, \; r(\mathbf{x}) = \langle \mathbf{x}, B\mathbf{x} \rangle$$

are simultaneously diagonalizable.

More precisely, there exists an invertible matrix S so that $S^T AS = \Lambda$ and $S^T BS = I$ and if $\mathbf{x} = S\mathbf{y}$ then

$$q(\mathbf{x}) = q(S\mathbf{y}) = \langle \mathbf{y}, \Lambda\mathbf{y} \rangle, \quad r(\mathbf{x}) = r(S\mathbf{y}) = \langle \mathbf{y}, \mathbf{y} \rangle$$

The diagonal matrix Λ consists of the generalized eigenvalues solutions of the problem $A\mathbf{v} = \lambda B\mathbf{v}$ and the columns of S are its generalized eigenvectors.

4. The Rayleigh's principle. The minimax theorem for the eigenvalues of a self-adjoint matrix

Eigenvalues of self-adjoint matrices are easy to calculate. This section shows how this is done using a minimization, or maximization procedure.

4.1. The Rayleigh's quotient.

Definition 33. *Let $A = A^*$ be a self-adjoint matrix.* **The Rayleigh's quotient** *is the function*

$$R(\mathbf{x}) = \frac{\langle \mathbf{x}, A\mathbf{x}\rangle}{\|\mathbf{x}\|^2}, \quad \text{for } \mathbf{x} \neq \mathbf{0}$$

Note that

$$R(\mathbf{x}) = \langle \frac{\mathbf{x}}{\|\mathbf{x}\|}, A\frac{\mathbf{x}}{\|\mathbf{x}\|}\rangle = \langle \mathbf{u}, A\mathbf{u}\rangle \quad \text{where } \mathbf{u} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

so in fact, it suffices to define the Rayleigh's quotient on unit vectors.

The set of unit vectors in $\mathbb{R}^n$ (or in $\mathbb{C}^n$), is called the $n-1$ dimensional sphere in $\mathbb{R}^n$ (or in $\mathbb{C}^n$):

$$S_F^{n-1} = \{\mathbf{u} \in F^n \,|\, \|\mathbf{u}\| = 1\}$$

For example, the sphere in $\mathbb{R}^2$ is the unit circle (it is a curve, it has dimension 1), the sphere in $\mathbb{R}^3$ is the unit sphere (it is a surface, it has dimension 2); for higher dimensions we need to use our imagination.

4.2. Extrema of the Rayleigh's quotient.

4.2.1. *Closed sets, bounded sets, compact sets.* You probably know very well the extreme value theorem for continuous function on the real line:

Theorem 34. The extreme value theorem in dimension one.
A functions $f(x)$ which is continuous on a closed and bounded interval $[a, b]$ has a maximum value (and a minimum value) on $[a, b]$.

To formulate an analogue of this theorem in higher dimensions we need to specify what we mean by a *closed* set and by a *bounded* set.

Definition 35. *A set S is called* **closed** *if it contains all its limit points: if a sequence of points in S, $\{\mathbf{x}_k\}_k \subset S$ converges, $\lim_{k\to\infty} = \mathbf{x}$, then the limit $\mathbf{x}$ is also in S.*

For example, the intervals $[2, 6]$ and $[2, +\infty)$ are closed in $\mathbb{R}$, but $[2, 6)$ is not closed. The closed unit disk $\{\mathbf{x} \in \mathbb{R}^2 \,|\, \|\mathbf{x}\| \leq 1\}$ is closed in $\mathbb{R}^2$, but the punctured disk $\{\mathbf{x} \in \mathbb{R}^2 \,|\, 0 < \|\mathbf{x}\| \leq 1\}$ or the open disk $\{\mathbf{x} \in \mathbb{R}^2 \,|\, \|\mathbf{x}\| < 1\}$ are not closed sets.

Definition 36. *A set S is called* **bounded** *if there is a number larger than all the lengths of the vectors in S: there is $M > 0$ so that $\|\mathbf{x}\| \leq M$ for all $\mathbf{x} \in S$.*

For example, the intervals $[2,6]$ and $[2,6)$ are bounded in $\mathbb{R}$, but $[2,+\infty)$ is not. The square $\{\mathbf{x} \in \mathbb{R}^2 \,|\, |x_1| < 1, \text{ and } |x_2| < 1\}$ is bounded in $\mathbb{R}^2$, but the vertical strip $\{\mathbf{x} \in \mathbb{R}^2 \,|\, |x_1| < 1\}$ is not.

Theorem 37. The extreme value theorem in finite dimensions.
A functions $f(x)$ which is continuous on a closed and bounded set S in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a maximum value (and a minimum value) on S.

In infinite dimensions Theorem 37 is not true in this form. A more stringent condition on the set S is needed to ensure existence of extreme values of continuous functions on S (the set must be *compact*).

It is intuitively clear (and rigorously proved in mathematical analysis) that any sphere in F^n is a closed and bounded set.

4.2.2. *Minimum and maximum of the Rayleigh's quotient.* The Rayleigh's quotient calculated on unit vectors is a quadratic polynomial, and therefore it is a continuous function on the unit sphere, and therefore

Proposition 38. *The Rayleigh's quotient has a maximum and a minimum.*

What happens if A is not self-adjoint? Recall that the quadratic form $\langle \mathbf{x}, A\mathbf{x}\rangle$ has the same value if we replace A by its self-adjoint part, $\frac{1}{2}(A+A^*)$, therefore the Rayleigh's quotient of A is the same as the Rayleigh's quotient of the self-adjoint part of A (information about A is lost).

The extreme values of the Rayleigh's quotient are linked to the eigenvalues of the self-adjoint matrix A. To see this, diagonalize the quadratic form $\langle \mathbf{x}, A\mathbf{x}\rangle$: consider a unitary matrix U which diagonalizes A:

$$U^*AU = \Lambda = diag(\lambda_1, \ldots, \lambda_n)$$

In the new coordinates $\mathbf{y} = U^*\mathbf{x}$ we have

$$\langle \mathbf{x}, A\mathbf{x}\rangle = \langle \mathbf{x}, U\Lambda U^*\mathbf{x}\rangle = \langle U^*\mathbf{x}, \Lambda U^*\mathbf{x}\rangle = \langle \mathbf{y}, \Lambda\mathbf{y}\rangle = \sum_{j=1}^{n} \lambda_j |y_j|^2$$

which together with $\|\mathbf{x}\| = \|U\mathbf{y}\| = \|\mathbf{y}\|$ give

$$R(\mathbf{x}) = R(U\mathbf{y}) = \frac{\sum_{j=1}^{n} \lambda_j |y_j|^2}{\|\mathbf{y}\|^2} = \sum_{j=1}^{n} \lambda_j \frac{|y_j|^2}{\|\mathbf{y}\|^2} \equiv R_U(\mathbf{y}) \tag{22}$$

Since A is self-adjoint, its eigenvalues λ_j are real; assume them ordered in an increasing sequence:

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

Then clearly

$$\sum_{j=1}^{n} \lambda_j |y_j|^2 \le \lambda_n \sum_{j=1}^{n} |y_j|^2 = \lambda_n \|\mathbf{y}\|^2$$

and

$$\sum_{j=1}^{n} \lambda_j |y_j|^2 \geq \lambda_1 \sum_{j=1}^{n} |y_j|^2 = \lambda_1 \|\mathbf{y}\|^2$$

therefore

$$\lambda_1 \leq R(\mathbf{x}) \leq \lambda_n \quad \text{for all } \mathbf{x} \neq \mathbf{0}$$

Equalities are attained since $R_U(\mathbf{e}_1) = \lambda_1$ and $R_U(\mathbf{e}_n) = \lambda_n$. Going to coordinates $\mathbf{x}$ minimum is attained for $\mathbf{x} = U\mathbf{e}_1 = \mathbf{v}_1$ = eigenvector corresponding to λ_1 since $R(\mathbf{v}_1) = R_U(\mathbf{e}_1) = \lambda_1$, and for $\mathbf{x} = U\mathbf{v}_n = \mathbf{v}_n$ = eigenvector corresponding to λ_n, maximum is attained since $R(\mathbf{v}_n) = R_U(\mathbf{e}_n) = \lambda_n$. This proves:

Theorem 39. *If A is a self-adjoint matrix then*

$$\max \frac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\|\mathbf{x}\|^2} = \lambda_n \textit{ the max eigenvalue of } A, \textit{ attained for } \mathbf{x} = \mathbf{v}_n$$

and

$$\min \frac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\|\mathbf{x}\|^2} = \lambda_1 \textit{ the min eigenvalue of } A, \textit{ attained for } \mathbf{x} = \mathbf{v}_1$$

As an important consequence in numerical calculations: the maximum eigenvalue of A can be found by solving a maximization problem, and the minimum eigenvalue - by a minimization problem.

4.3. **The minimax principle.** Reducing the dimension of A we can find all the eigenvalues, one by one. This reduction of the dimension is done using:

4.3.1. *Spitting of the space under the action of a self-adjoint matrix.* Any matrix leaves invariant its eigenspaces. Since eigenvectors of a self-adjoint matrix A form an orthogonal set, then self-adjoint matrices leaves invariant their orthogonal complement as well:

Remark. *Let A be any $n \times n$ self-adjoint matrix.*

a) The vector space F^n splits

$$F^n = \oplus_{\lambda \in \sigma(A)} V_\lambda$$

where each eigenspace V_λ is invariant under A, and so is its orthogonal complement, which equals the direct sum of all the other eigenspaces:

$$V_\lambda^\perp = \oplus_{\lambda' \in \sigma(A), \lambda' \neq \lambda} \quad V_{\lambda'}$$

b) If $\mathbf{v}$ is an eigenvector then A leaves invariant $Sp(\mathbf{v})$ and $Sp(\mathbf{v})^\perp$.

Proof:

Only part b) needs a proof.

$A(Sp(\mathbf{v})) \subset Sp(\mathbf{v})$ because if $\mathbf{x} \in Sp(\mathbf{v})$ then $\mathbf{x} = x\mathbf{v}$ therefore $A\mathbf{x} = cA\mathbf{v} = c\lambda\mathbf{v} \in Sp(\mathbf{v})$.

$A(Sp(\mathbf{v})^\perp) \subset Sp(\mathbf{v})^\perp$ because if $\mathbf{y} \in Sp(\mathbf{v})^\perp$ then this means that $\langle \mathbf{y}, \mathbf{v} \rangle = 0$. Then $\langle A\mathbf{y}, \mathbf{v} \rangle = \langle \mathbf{y}, A^*\mathbf{v} \rangle = \langle \mathbf{y}, A\mathbf{v} \rangle = \langle \mathbf{y}, \lambda\mathbf{v} \rangle = \lambda \langle \mathbf{y}, \mathbf{v} \rangle = 0$. $\square$

4.3.2. *Minimax and maximin.* We saw that $\max R(\mathbf{x}) = \lambda_n = R(\mathbf{v}_n)$. Then the matrix A, as a linear transformation of the $n-1$ dimensional vector space $Sp(\mathbf{v}_n)^\perp$ to itself has its largest eigenvalue λ_{n-1} (we reduced the dimension!). Then

$$\max_{\mathbf{x}\in Sp(\mathbf{v}_n)^\perp} R(\mathbf{x}) = \lambda_{n-1} \quad \text{is attained for } \mathbf{x} = \mathbf{v}_{n-1}$$

Note that the statement $\mathbf{x} \in Sp(\mathbf{v}_n)^\perp$ can be formulated as the constraint $\langle \mathbf{x}, \mathbf{v}_n\rangle = 0$:

$$\max_{\langle \mathbf{x},\mathbf{v}_n\rangle=0} R(\mathbf{x}) = \lambda_{n-1}$$

We can do even better: we can obtain λ_{n-1} without knowing $\mathbf{v}_n$ or λ_n. To achieve this, subject $\mathbf{x}$ to *any* constraint: $\langle \mathbf{x}, \mathbf{z}\rangle = 0$ for some $\mathbf{z} \neq \mathbf{0}$.

It is easier to see what happens in coordinates $\mathbf{y} = U^*\mathbf{x}$ in which A is diagonal. The constraint $\langle \mathbf{x}, \mathbf{z}\rangle = 0$ is equivalent to $\langle \mathbf{y}, \mathbf{w}\rangle = 0$ where $\mathbf{w} = U\mathbf{z}$ is some nonzero vector. On one hand, obviously

$$\max_{\mathbf{y}:\, \langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) \leq \lambda_n$$

$$\max_{\mathbf{y}:\, \langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) \geq \lambda_{n-1} \tag{23}$$

which implies that

$$\min_{\mathbf{w}\neq\mathbf{0}} \max_{\mathbf{y}:\, \langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) \geq \lambda_{n-1} \tag{24}$$

We now argue that equality is atained in (24) for special $\mathbf{w}$ and $\mathbf{y}$. Indeed, consider a nonzero vector $\tilde{\mathbf{y}} = (0, \dots, 0, y_{n-1}, y_n)^T$ with $\langle \tilde{\mathbf{y}}, \mathbf{w}\rangle = 0$. ($\tilde{\mathbf{y}}$ is easy to find: if $w_n \neq 0$ take $y_{n-1} = 1$ and $y_n = -w_{n-1}/w_n$, and if $w_n = 0$ take $y_{n-1} = 0,\ y_n = 1$).

Using formula (22)

$$R_U(\tilde{\mathbf{y}}) = \frac{\lambda_{n-1}|y_{n-1}|^2 + \lambda_n|y_n|^2}{|y_{n-1}|^2 + |y_n|^2} \geq \frac{\lambda_{n-1}|y_{n-1}|^2 + \lambda_{n-1}|y_n|^2}{|y_{n-1}|^2 + |y_n|^2} = \lambda_{n-1}$$

Since for $\mathbf{w} = \mathbf{e}_n$ we have equality:

$$\max_{\mathbf{y}:\, \langle \mathbf{y},\mathbf{e}_n\rangle=0} R_U(\mathbf{y}) = \lambda_{n-1}$$

then in (24) there is equality

$$\min_{\mathbf{w}\neq\mathbf{0}} \max_{\mathbf{y}:\, \langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) = \lambda_{n-1}$$

In a similar way it is shown that λ_{n-2} is obtained by a minimum-maximum process, but with two constraints:

$$\min_{\mathbf{w}_1,\mathbf{w}_2\neq\mathbf{0}} \max_{\substack{\langle \mathbf{y},\mathbf{w}_1\rangle = 0 \\ \langle \mathbf{y},\mathbf{w}_2\rangle = 0}} R_U(\mathbf{y}) = \lambda_{n-2} \tag{25}$$

Indeed, consider a nonzero vector $\tilde{\mathbf{y}} = (0, \ldots, 0, y_{n-2}, y_{n-1}, y_n)^T$ satisfying $\langle \tilde{\mathbf{y}}, \mathbf{w}_1 \rangle = 0$ and $\langle \tilde{\mathbf{y}}, \mathbf{w}_2 \rangle = 0$. Then in formula (22)

$$R_U(\tilde{\mathbf{y}}) = \frac{\lambda_{n-2}|y_{n-2}|^2 + \lambda_{n-1}|y_{n-1}|^2 + \lambda_n|y_n|^2}{|y_{n-2}|^2 + |y_{n-1}|^2 + |y_n|^2}$$

$$\geq \frac{\lambda_{n-2}|y_{n-2}|^2 + \lambda_{n-2}|y_{n-1}|^2 + \lambda_{n-2}|y_n|^2}{|y_{n-2}|^2 + |y_{n-1}|^2 + |y_n|^2} = \lambda_{n-2}$$

which shows that

$$\max_{\substack{\langle \mathbf{y}, \mathbf{w}_1 \rangle = 0 \\ \langle \mathbf{y}, \mathbf{w}_2 \rangle = 0}} R_U(\mathbf{y}) \geq \lambda_{n-2} \tag{26}$$

Since for $\mathbf{w}_1 = \mathbf{e}_n$ and $\mathbf{w}_2 = \mathbf{e}_{n-1}$ we have equality in (26), this implies (25).

Step by step, adding one extra constraint, the minimax procedure yields the next largest eigenvalue.

Going back to the variable $\mathbf{x}$ it is found that:

Theorem 40. The minimax principle

Let A be a self-adjoint matrix, with its eigenvalues numbered in an increasing sequence:

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

corresponding to the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Then its Rayleigh's quotient

$$R(\mathbf{x}) = \frac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\|\mathbf{x}\|^2}$$

satisfies

$$\max_{\mathbf{x} \neq 0} R(\mathbf{x}) = \lambda_n$$

$$\min_{\mathbf{z} \neq \mathbf{0}} \max_{\langle \mathbf{x}, \mathbf{z} \rangle = 0} R(\mathbf{x}) = \lambda_{n-1}$$

$$\min_{\mathbf{z}_1, \mathbf{z}_2 \neq \mathbf{0}} \max_{\substack{\langle \mathbf{x}, \mathbf{z}_1 \rangle = 0 \\ \langle \mathbf{x}, \mathbf{z}_2 \rangle = 0}} R(\mathbf{x}) = \lambda_{n-2}$$

and in general

$$\min_{\mathbf{z}_1, \ldots \mathbf{z}_k \neq \mathbf{0}} \max_{\substack{\langle \mathbf{x}, \mathbf{z}_1 \rangle = 0 \\ \vdots \\ \langle \mathbf{x}, \mathbf{z}_k \rangle = 0}} R(\mathbf{x}) = \lambda_{n-k}, \qquad k = 1, 2, \ldots, n-1$$

Remark. Sometimes the minimax principle is formulated as

$$\min_{V_j} \max_{\mathbf{x} \in V_j} R(\mathbf{x}) = \lambda_j, \quad j = 1, \ldots, n$$

where V_j denotes an arbitrary subspace of dimension j.

The two formulations are equivalent since the set

$$V_{n-k} = \{\mathbf{x} \,|\, \langle \mathbf{x}, \mathbf{z}_1 \rangle = 0, \ldots, \langle \mathbf{x}, \mathbf{z}_k \rangle = 0\}$$

is a vector space of dimension $n-k$ if $\mathbf{z}_1, \ldots, \mathbf{z}_k$ are linearly independent.

A similar construction starting with the lowest eigenvalue produces:

Theorem 41. The maximin principle
Under the assumptions of Theorem 40

$$\min_{\mathbf{x} \neq 0} R(\mathbf{x}) = \lambda_1$$

$$\max_{\mathbf{z} \neq \mathbf{0}} \min_{\langle \mathbf{x}, \mathbf{z} \rangle = 0} R(\mathbf{x}) = \lambda_2$$

and in general

$$\max_{\mathbf{z}_1, \ldots \mathbf{z}_k \neq \mathbf{0}} \quad \min_{\substack{\langle \mathbf{x}, \mathbf{z}_1 \rangle = 0 \\ \vdots \\ \langle \mathbf{x}, \mathbf{z}_k \rangle = 0}} R(\mathbf{x}) = \lambda_{k+1}, \qquad k = 1, 2, \ldots, n-1$$

4.4. **The minimax principle for the generalized eigenvalue problem.** Suppose $\lambda_1 \leq \lambda_1 \leq \ldots \leq \lambda_n$ are eigenvalues for the problem

$$A\mathbf{v} = \lambda B\mathbf{v}, \quad A \text{ symmetric}, \ B \text{ positive definite} \tag{27}$$

It was seen in §3.8 that if $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is the matrix whose columns are the generalized eigenvectors of the problem (27), then both matrices A and B are diagonalized using a congruence transformation: $S^T AS = \Lambda$ and $S^T BS = I$.

Defining

$$R(\mathbf{x}) = \frac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\langle \mathbf{x}, B\mathbf{x} \rangle}$$

it is found that in coordinates $\mathbf{x} = S\mathbf{y}$:

$$R(\mathbf{x}) = R(S\mathbf{y}) = \frac{\langle S\mathbf{y}, AS\mathbf{y} \rangle}{\langle S\mathbf{y}, BS\mathbf{y} \rangle} = \frac{\langle \mathbf{y}, S^T AS\mathbf{y} \rangle}{\langle \mathbf{y}, S^T BS\mathbf{y} \rangle} = \frac{\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2}{y_1^2 + \ldots + y_n^2}$$

and therefore

$$\max R(\mathbf{x}) = \lambda_n, \quad \min R(\mathbf{x}) = \lambda_1$$