and

$$\sum_{j=1}^{n} \lambda_j |y_j|^2 \geq \lambda_1 \sum_{j=1}^{n} |y_j|^2 = \lambda_1 \|\mathbf{y}\|^2$$

therefore

$$\lambda_1 \leq R(\mathbf{x}) \leq \lambda_n \quad \text{for all } \mathbf{x} \neq \mathbf{0}$$

Equalities are attained since $R_U(\mathbf{e}_1) = \lambda_1$ and $R_U(\mathbf{e}_n) = \lambda_n$. Going to coordinates $\mathbf{x}$ minimum is attained for $\mathbf{x} = U\mathbf{e}_1 = \mathbf{u}_1$ = eigenvector corresponding to λ_1 since $R(\mathbf{u}_1) = R_U(\mathbf{e}_1) = \lambda_1$, and for $\mathbf{x} = U\mathbf{e}_n = \mathbf{u}_n$ = eigenvector corresponding to λ_n, maximum is attained since $R(\mathbf{u}_n) = R_U(\mathbf{e}_n) = \lambda_n$. This proves:

Theorem 39. *If A is a self-adjoint matrix then*

$$\max \frac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\|\mathbf{x}\|^2} = \lambda_n \textit{ the max eigenvalue of } A, \textit{ attained for } \mathbf{x} = \mathbf{u}_n$$

and

$$\min \frac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\|\mathbf{x}\|^2} = \lambda_1 \textit{ the min eigenvalue of } A, \textit{ attained for } \mathbf{x} = \mathbf{u}_1$$

As an important consequence in numerical calculations: the maximum eigenvalue of A can be found by solving a maximization problem, and the minimum eigenvalue - by a minimization problem.

4.3. **The minimax principle.** Reducing the dimension of A we can find all the eigenvalues, one by one. Consider the eigenvalues (24) of A and the corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ which form an orthonormal basis: $F^n = \oplus_{j=1}^{n} F\mathbf{u}_j$.

We saw that $\max R(\mathbf{x}) = \lambda_n = R(\mathbf{u}_n)$. The subspace $Sp(\mathbf{u}_n)$ and its orthogonal complement $Sp(\mathbf{u}_n)^\perp = \oplus_{j=1}^{n-1} Sp(\mathbf{u}_j)$ are invariant under A.

Consider A as a linear transformation of the $n-1$ dimensional vector space $Sp(\mathbf{u}_n)^\perp$ to itself: its eigenvalues are $\lambda_1, \ldots, \lambda_{n-1}$, the largest being λ_{n-1}. We reduced the dimension!

Using Theorem 39 for A as a linear transformation on the vector space $Sp(\mathbf{u}_n)^\perp$ it follows that

$$\max_{\mathbf{x} \in Sp(\mathbf{u}_n)^\perp} R(\mathbf{x}) = \lambda_{n-1} \quad \text{is attained for } \mathbf{x} = \mathbf{u}_{n-1} \tag{25}$$

The statement $\mathbf{x} \in Sp(\mathbf{u}_n)^\perp$ can be formulated as the constraint $\langle \mathbf{x}, \mathbf{u}_n \rangle = 0$:

$$\max_{\mathbf{x}: \langle \mathbf{x}, \mathbf{u}_n \rangle = 0} R(\mathbf{x}) = \lambda_{n-1}$$

We can do even better: we can *obtain λ_{n-1} without knowing $\mathbf{u}_n$ or λ_n*. To achieve this, subject $\mathbf{x}$ to *any* constraint: $\langle \mathbf{x}, \mathbf{z} \rangle = 0$ for some $\mathbf{z} \neq \mathbf{0}$.

It is easier to see what happens in coordinates $\mathbf{y} = U^*\mathbf{x}$ in which A is diagonal. The constraint $\langle \mathbf{x}, \mathbf{z} \rangle = 0$ is equivalent to $\langle \mathbf{y}, \mathbf{w} \rangle = 0$ where $\mathbf{w} = U\mathbf{z}$ is some nonzero vector.

Step I. We have

$$\max_{\mathbf{y}:\,\langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) \geq \lambda_{n-1} \qquad \text{for all } \mathbf{w} \neq \mathbf{0} \tag{26}$$

since there is some nonzero vector $\tilde{\mathbf{y}}$ belonging to both the $n-1$ dimensional subspace $\{\mathbf{y} : \langle \mathbf{y}, \mathbf{w}\rangle = 0\}$ and the two dimensional subspace $F\mathbf{e}_{n-1} \oplus F\mathbf{e}_n$. (Such a vector is easy to find: $\tilde{\mathbf{y}} = (0, \ldots, 0, y_{n-1}, y_n)^T$ with $\langle \tilde{\mathbf{y}}, \mathbf{w}\rangle = 0$; if $w_n \neq 0$ take $y_{n-1} = 1$ and $y_n = -w_{n-1}/w_n$, and if $w_n = 0$ take $y_{n-1} = 0,\ y_n = 1$). Using formula (23)

$$R_U(\tilde{\mathbf{y}}) = \frac{\lambda_{n-1}|y_{n-1}|^2 + \lambda_n|y_n|^2}{|y_{n-1}|^2 + |y_n|^2} \geq \frac{\lambda_{n-1}|y_{n-1}|^2 + \lambda_{n-1}|y_n|^2}{|y_{n-1}|^2 + |y_n|^2} = \lambda_{n-1} \tag{27}$$

proving (26).

Step II. Inequality (26) implies that

$$\min_{\mathbf{w}\neq\mathbf{0}} \max_{\mathbf{y}:\,\langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) \geq \lambda_{n-1} \tag{28}$$

Step III. We now show that equality is attained in (28) for special $\mathbf{w}$.

For $\mathbf{w} = \mathbf{e}_n$ we have, by (25),

$$\max_{\mathbf{y}:\,\langle \mathbf{y},\mathbf{e}_n\rangle=0} R_U(\mathbf{y}) = \lambda_{n-1} \quad \text{attained for } \mathbf{y} = \mathbf{e}_{n-1}$$

hence in (28) there is equality

$$\min_{\mathbf{w}\neq\mathbf{0}} \max_{\mathbf{y}:\,\langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) = \lambda_{n-1}$$

In a similar way it is shown that λ_{n-2} is obtained by a minimum-maximum process, but with two constraints:

$$\min_{\mathbf{w}_1,\mathbf{w}_2\neq\mathbf{0}} \max_{\substack{\langle \mathbf{y},\mathbf{w}_1\rangle = 0 \\ \langle \mathbf{y},\mathbf{w}_2\rangle = 0}} R_U(\mathbf{y}) = \lambda_{n-2} \tag{29}$$

Indeed, consider a nonzero vector $\tilde{\mathbf{y}} = (0, \ldots, 0, y_{n-2}, y_{n-1}, y_n)^T$ satisfying $\langle \tilde{\mathbf{y}}, \mathbf{w}_1\rangle = 0$ and $\langle \tilde{\mathbf{y}}, \mathbf{w}_2\rangle = 0$. Then in formula (23)

$$\begin{aligned} R_U(\tilde{\mathbf{y}}) &= \frac{\lambda_{n-2}|y_{n-2}|^2 + \lambda_{n-1}|y_{n-1}|^2 + \lambda_n|y_n|^2}{|y_{n-2}|^2 + |y_{n-1}|^2 + |y_n|^2} \\ &\geq \frac{\lambda_{n-2}|y_{n-2}|^2 + \lambda_{n-2}|y_{n-1}|^2 + \lambda_{n-2}|y_n|^2}{|y_{n-2}|^2 + |y_{n-1}|^2 + |y_n|^2} = \lambda_{n-2} \end{aligned}$$

which shows that

$$\max_{\substack{\langle \mathbf{y},\mathbf{w}_1\rangle = 0 \\ \langle \mathbf{y},\mathbf{w}_2\rangle = 0}} R_U(\mathbf{y}) \geq \lambda_{n-2} \tag{30}$$

Since for $\mathbf{w}_1 = \mathbf{e}_n$ and $\mathbf{w}_2 = \mathbf{e}_{n-1}$ we have equality in (30), and this implies (29).

Step by step, adding one extra constraint, the minimax procedure yields the next largest eigenvalue.

Going back to the variable $\mathbf{x}$ it is found that:

Theorem 40. The minimax principle

Let A be a self-adjoint matrix, with its eigenvalues numbered in an increasing sequence:

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

corresponding to the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Then its Rayleigh's quotient

$$R(\mathbf{x}) = \frac{\langle \mathbf{x}, A\mathbf{x} \rangle}{\|\mathbf{x}\|^2}$$

satisfies

$$\max_{\mathbf{x} \neq 0} R(\mathbf{x}) = \lambda_n$$

$$\min_{\mathbf{z} \neq \mathbf{0}} \max_{\langle \mathbf{x}, \mathbf{z} \rangle = 0} R(\mathbf{x}) = \lambda_{n-1}$$

$$\min_{\mathbf{z}_1, \mathbf{z}_2 \neq \mathbf{0}} \max_{\substack{\langle \mathbf{x}, \mathbf{z}_1 \rangle = 0 \\ \langle \mathbf{x}, \mathbf{z}_2 \rangle = 0}} R(\mathbf{x}) = \lambda_{n-2}$$

and in general

$$\min_{\mathbf{z}_1, \ldots \mathbf{z}_k \neq \mathbf{0}} \max_{\substack{\langle \mathbf{x}, \mathbf{z}_1 \rangle = 0 \\ \vdots \\ \langle \mathbf{x}, \mathbf{z}_k \rangle = 0}} R(\mathbf{x}) = \lambda_{n-k}, \qquad k = 1, 2, \ldots, n-1$$

Remark. Sometimes the minimax principle is formulated as

$$\min_{V_j} \max_{\mathbf{x} \in V_j} R(\mathbf{x}) = \lambda_j, \quad j = 1, \ldots, n$$

where V_j denotes an arbitrary subspace of dimension j.

The two formulations are equivalent since the set

$$V_{n-k} = \{\mathbf{x} \,|\, \langle \mathbf{x}, \mathbf{z}_1 \rangle = 0, \ldots, \langle \mathbf{x}, \mathbf{z}_k \rangle = 0\}$$

is a vector space of dimension $n-k$ if $\mathbf{z}_1, \ldots, \mathbf{z}_k$ are linearly independent.

A similar construction starting with the lowest eigenvalue produces:

Theorem 41. The maximin principle

Under the assumptions of Theorem 40

$$\min_{\mathbf{x} \neq 0} R(\mathbf{x}) = \lambda_1$$

$$\max_{\mathbf{z} \neq \mathbf{0}} \min_{\langle \mathbf{x}, \mathbf{z} \rangle = 0} R(\mathbf{x}) = \lambda_2$$

and in general

$$\max_{\mathbf{z}_1,\ldots\mathbf{z}_k\neq\mathbf{0}} \quad \min_{\substack{\langle \mathbf{x},\mathbf{z}_1\rangle = 0 \\ \vdots \\ \langle \mathbf{x},\mathbf{z}_k\rangle = 0}} \quad R(\mathbf{x}) = \lambda_{k+1}, \qquad k = 1, 2, \ldots, n-1$$

4.4. **The minimax principle for the generalized eigenvalue problem.** Suppose $\lambda_1 \leq \lambda_1 \leq \ldots \leq \lambda_n$ are eigenvalues for the problem

$$A\mathbf{v} = \lambda B\mathbf{v}, \quad A \text{ symmetric}, \ B \text{ positive definite} \tag{31}$$

It was seen in §3.8 that if $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is the matrix whose columns are the generalized eigenvectors of the problem (31), then both matrices A and B are diagonalized using a congruence transformation: $S^T AS = \Lambda$ and $S^T BS = I$.

Defining

$$R(\mathbf{x}) = \frac{\langle \mathbf{x}, A\mathbf{x}\rangle}{\langle \mathbf{x}, B\mathbf{x}\rangle}$$

it is found that in coordinates $\mathbf{x} = S\mathbf{y}$:

$$R(\mathbf{x}) = R(S\mathbf{y}) = \frac{\langle S\mathbf{y}, AS\mathbf{y}\rangle}{\langle S\mathbf{y}, BS\mathbf{y}\rangle} = \frac{\langle \mathbf{y}, S^T AS\mathbf{y}\rangle}{\langle \mathbf{y}, S^T BS\mathbf{y}\rangle} = \frac{\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2}{y_1^2 + \ldots + y_n^2}$$

and therefore

$$\max R(\mathbf{x}) = \lambda_n, \quad \min R(\mathbf{x}) = \lambda_1$$

5. Singular Value Decomposition

5.1. **Rectangular matrices.**

For rectangular matrices M the notions of eigenvalue/vector cannot be defined. However, the products MM^* and/or M^*M (which are square, even self-adjoint, and even positive semi-definite matrices) carry a lot of information about M. The first result is that they have the same nonzero eigenvalues:

Proposition 42. *Let M be an $m \times n$ matrix. The matrices MM^* and M^*M are positive semi-definite. Moreover, they have the same nonzero eigenvalues (with the same multiplicity).*

Moreover, let $\lambda_1, \dots, \lambda_r$ be the positive eigenvalues. If $M^*M\mathbf{v}_j = \lambda_j \mathbf{v}_j$ with $\lambda_j > 0$ and $\mathbf{v}_1, \dots, \mathbf{v}_r$ an orthonormal set, then $MM^*\mathbf{u}_j = \lambda_j \mathbf{u}_j$ for $\mathbf{u}_j = \frac{1}{\sqrt{\lambda_j}} M\mathbf{v}_j$ and $\mathbf{u}_1, \dots, \mathbf{u}_r$ is an orthonormal set.

Proof. The two matrices obviously self-adjoint and are positive semi-definite since $\langle \mathbf{x}, M^*M\mathbf{x}\rangle = \langle M\mathbf{x}, M\mathbf{x}\rangle \geq 0$ and $\langle \mathbf{x}, MM^*\mathbf{x}\rangle = \langle M^*\mathbf{x}, M^*\mathbf{x}\rangle \geq 0$.

Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be an orthonormal set of eigenvectors of M^*M, the first r corresponding to nonzero eigenvalues: $M^*M\mathbf{v}_j = \lambda_j \mathbf{v}_j$ with $\lambda_j > 0$, for $j = 1, \dots, r$ and $M^*M\mathbf{v}_j = \mathbf{0}$ for $j > r$.

Applying M we discover that $MM^*M\mathbf{v}_j = \lambda_j M\mathbf{v}_j$ with $\lambda_j > 0$, for $j = 1, \dots, r$ and $MM^*M\mathbf{v}_j = \mathbf{0}$ for $j > r$ which would mean that $M\mathbf{v}_j$ are eigenvectors to MM^* corresponding to the eigenvalue λ_j provided we ensure that $M\mathbf{v}_j \neq \mathbf{0}$. This is certainly true for all $j = 1, \dots, r$, since $\|M\mathbf{v}_j\|^2 = \langle M\mathbf{v}_j, M\mathbf{v}_j\rangle = \langle \mathbf{v}_j, M^*M\mathbf{v}_j\rangle = \langle \mathbf{v}_j, \lambda_j \mathbf{v}_j\rangle = \lambda_j \neq 0$ for $j \leq r$.

On the other hand, note that all $M\mathbf{v}_1, \dots, M\mathbf{v}_r$ are mutually orthogonal, since $\langle M\mathbf{v}_j, M\mathbf{v}_i\rangle = \langle \mathbf{v}_j, M^*M\mathbf{v}_i\rangle = \lambda_i \delta_{ij}$ so $M\mathbf{v}_j \perp M\mathbf{v}_i$ for all $i \neq j \leq r$, and $\|M\mathbf{v}_j\|^2 = \lambda_j$. Therefore, all the nonzero eigenvalues of M^*M are also eigenvalues for MM^*, and with the same multiplicity, and with corresponding unit eigenvectors $\mathbf{u}_j = \frac{1}{\sqrt{\lambda_j}} M\mathbf{v}_j$, $j = 1, \dots, r$.

The same argument can be applied replacing M by M^*, showing that indeed, MM^* and M^*M have the same nonzero eigenvalues and with the same multiplicity. $\square$

Another result is that the Euclidian norm of M can be read from the maximal eigenvalue of MM^* (or M^*M):

$$\sum_{i,j} |M_{ij}|^2 = \text{the max eigenvalue of } M^*M = \text{the max eigenvalue of } MM^*$$

(We will discuss more later.)

5.2. **The SVD theorem.**

Let M be an $m \times n$ matrix. We are going to bring it as close to a diagonal form as humanly possible. Namely, we are going to write it as $M = U\Sigma V^*$ where Σ is a diagonal $m \times n$ matrix, and U