III. THE DUAL SPACE

This is a linear space, of dimension \( n + 1 \) (you may wish to check!).

As in §1.6.1, we find the basis \( F_0, \ldots, F_n \) of \( V \) for which \( E_{s_0}, \ldots, E_{s_n} \) is the dual basis. (It is quite clear that these evaluations form a basis for \( V \) since any function is completely determined by its values at the sample points.) The piecewise linear function \( F_k \) must satisfy \( (E_{s_j}, F_k) = F_k(s_j) = 0 \) for all \( j \neq k \), and \( F_k(s_k) = 1 \). Therefore \( F_k \) is a "tent" function which is zero on \([s_{j-1}, s_j]\) and \([s_j, s_{j+1}]\) and whose graph joins by a segment the points \((s_{j-1}, 0)\) and \((s_j, 1)\) and by a segment the points \((s_j, 1)\) and \((s_{j+1}, 0)\).

Then

\[
\psi(t) = \sum_{k=0}^{n} y_k F_k(t)
\]

is the interpolating function in \( V \) for the data (6).

1.6.3. Band-limited interpolation. Another type of interpolating function is a superposition of oscillations. We assume here the samples are taken over an interval which we take to be \( 2\pi \) (for simplicity of formulas).

We look for interpolating functions in the linear space spanned by the linearly independent functions \( 1, \sin(kt), \cos(kt), k = 1, \ldots, N \):

\[
\mathcal{B}_N = \left\{ f \mid f(x) = \sum_{k=0}^{N} a_k \cos(kt) + \sum_{k=1}^{N} b_k \sin(kt), a_k, b_k \in \mathbb{R} \right\}
\]

Since this an \( 2N + 1 \) dimensional space, we need an equal number of samples: \( s_0, s_1, \ldots, s_{2N} \in [0, 2\pi] \).

As in the examples before, we look for a basis \( S_k, k = 0, \ldots, 2N \) of \( \mathcal{B}_N \) so that the evaluation functionals \( E_{s_k} \) from its dual basis.\(^1\) The function \( S_k \) must satisfy \( S_k(x_j) = 0 \) for all \( j \neq k \). By analogy with the Lagrange formula for polynomial interpolation a first attempt may be to think of \( S_k \) as a scalar multiple of the product of all \( \sin(t - s_j) \) with \( j \neq k \); however this does not work because this product belongs to \( \mathcal{B}_{2N} \) rather than \( \mathcal{B}_N \). What works however is to take \( S_k \) as a scalar multiple of the product of all \( \sin^2(t - s_j) \) with \( j \neq k \).

Indeed, first note that this product belongs to \( \mathcal{B}_N \). There is an even number of factors in the product, and a product of any two of them is \( \sin^2(t - s_j) \sin^2(t - s_i) = \cos^2 \left( \frac{t}{2} - \frac{s_i + s_j}{2} \right) \in \mathcal{B}_N \). Thus \( S_k \) is the product of \( \tilde{N} \) terms of the form \( a_k + b_k \sin t + c_k \cos t \) which is known to have the form of functions in \( \mathcal{B}_N \) (e.g. \( 2 \sin x \cos x = \sin(2x) \), \( 2 \cos^2 x = 1 + \cos(2x) \), \( 4 \cos^3 x = \cos(3x) + \cos x \) etc.). Normalizing to ensure that

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\(^1\)In this example we will not prove that the evaluations form a basis, rather attempt to produce the basis for \( \mathcal{B}_N \); we will succeed, proving that the evaluations were a basis indeed.
$S_k(x_k) = 1$ we obtain the formula

$$S_k(t) = \prod_{j=0, j\neq k}^{2N} \frac{\sin \frac{1}{2}(t - s_j)}{\sin \frac{1}{2}(s_k - s_j)}$$

and the interpolating function is

$$\psi(t) = \sum_{k=0}^{2N} y_k S_k(t)$$

1.7. The bidual. The dual $V'$ is a vector space, therefore it has a dual $V''$, called the bidual of $V$. It turns out that the bidual can be naturally identified with $V$ by the pairing $(\phi, x)$, which can be interpreted either as $\phi$ acting on $x$, or as $x$ acting on $\phi$:

**Theorem 7.** The bidual $V''$ is isomorphic to $V$.

*Why:* formalizing the intuitive argument above, let

$$V'' = \{ L : V' \to F \mid L \text{ linear} \}$$

and define $T : V \to V''$ as for each $x \in V$ let $Tx \in V''$ be defined as $(Tx)\phi = (\phi, x)$. Clearly $T$ is linear. To show $T$ is 1-to-1, assume that $Tx = 0$ for all $x \in V$. This means that $(Tx)\phi = (\phi, x) = 0$ for all $\phi \in V'$. It follows that $x = 0$, which is interesting in itself (it shows that linear functionals “separate” the elements of $V$), hence we state and prove separately, as Lemma 8 below. Once we show that $T$ is 1-to-1, since $\dim V = \dim V' = \dim V''$, then $T$ is also onto, hence it is an isomorphism. □.

**Lemma 8.** If $x \in V$ is so that $(\phi, x) = 0$ for all $\phi \in V'$, then $x = 0$.

To prove the Lemma, let $v_1, \ldots, v_n$ of a basis of $V$; then $x = \sum_j x_j v_j$ and we have in particular, $(v_k', x) = x_k = 0$ for all $k$, therefore $x = 0$.

*Note* that the construction of the isomorphism of Theorem 7 relies on no particular basis: this is called a *natural isomorphism*.

1.8. Orthogonality. The construction of hyperplanes suggests that the dual of a vector space can be used to define an orthogonality-like relation. For example,

**Definition 9.** If $U$ is a subspace of a vector space $V$ the annihilator of $U$ is the set

$$U^\perp = \{ \phi \in V' \mid (\phi, u) = 0, \text{ for all } u \in U \}$$

It can be checked that the annihilator $U^\perp$ is a subspace of $V'$. In absence of an inner product, the annihilator works quite well.
1.9. The transpose.

Warning: change of notation. Consistent notations are useful for helping the thought process focus on the essential features rather than mere names, and these notes have tried to use consistent notations. However...

Up to now we used consistently the letter $T$ to denote linear transformations. At this point we are going to encounter the transpose of matrices, denoted by a superscript $T$. To avoid collisions of notations, from now on linear transformations will be denoted by the letter $L$.

**Definition 10.** If $L : U \to V$ is a linear transformation between two vector spaces $U, V$ over the scalar field $F$ then the **transpose transformation** is the linear transformation $L^T : V^T \to U^T$ defined by

$$(8) \quad (L^T \phi, x) = (\phi, Lx) \quad \text{for all } x \in U, \phi \in V^T$$

As the words suggest:

**Theorem 11.** If $M$ is the matrix of $L : U \to V$ in the bases $B_U, B_V$ then $M^T$ is the matrix of $L^T$ in the dual bases $B^T_V, B^T_U$.

**Proof.** Since $M$ is the matrix of $L$ we have $L u_k = \sum_i M_{ik} v_i$. To calculate $L^T v_j$ we apply it to the vectors $x$:

$$(L^T v_j, x) = (L^T v_j, \sum_k x_k u_k) = \sum_k x_k (L^T v_j, u_k) = \sum_k x_k (v_j', Lu_k)$$

$$= \sum_k x_k (v_j', \sum_i M_{ik} v_i) = \sum_k \sum_i x_k M_{ik} (v_j', v_i) = \sum_k \sum_i x_k M_{ik} \delta_{ij}$$

$$= \sum_k x_k M_{jk} = \sum_k M_{jk} (u'_k, x) = (\sum_k M_{jk} u'_k, x) \text{ for all } x$$

which shows that $L^T v_j = \sum_k M_{jk} u'_k$, which proves the theorem. $\square$

1.9.1. The four fundamental spaces of a matrix. Consider an $m \times n$ matrix $M$ with entries in $F$ ($F = \mathbb{R}$ or $\mathbb{C}$) and its transpose $M^T$. To rewrite (8) in matrix notation it is sometimes preferred to replace the row vectors $\phi$ by transposes of (usual) column vectors, $y^T$. Relation (8) then reads $(M^T y)^T x = y^T M x$ for all $x \in \mathbb{R}^n$ (or $\mathbb{C}^n$) and all $y \in \mathbb{R}^m$ (respectively $\mathbb{C}^M$), a relation which is clear bases on the basic property that $(AB)^T = B^T A^T$ for any two matrices $A, B$ for which the multiplication makes sense.

Recall that we defined:

$\mathcal{R}(M) = \{ y \in F^m \mid Mx = y \text{ for some } x \in F^n \}$ the **column space** of $M$

$\mathcal{N}(M) = \{ x \in F^n \mid Mx = 0 \}$ the **null space** of $M$

Since the rows of $M^T$ are the columns of $M$, then clearly the row space of $M^T$ equals the column space of $M$, and column space of $M^T$ equals the row space of $M$:

$\mathcal{R}(M^T) = \text{the row space of } M$.
Define 
\[ N(M^T) = \{ y \in F^m \mid M^T y = 0 \} \] the left null space of \( M \).

Note that \( M^T y = 0 \) is equivalent to \( y^T M = 0 \), justifying the name of left null space.

Recall that \( \dim \mathcal{R}(M) + \dim \mathcal{N}(M) = n \) and using this for \( M^T \), we obtain \( \dim \mathcal{R}(M^T) + \dim \mathcal{N}(M^T) = m \).

Recall that \( \dim \mathcal{R}(M) = \text{rank}(M) \) and the rank of a matrix is the order of the largest nonzero minor; then \( \text{rank}(M) = \text{rank}(M^T) \).

Summarizing:

**Theorem 12. The fundamental theorem of linear algebra**

Let \( M \) be an \( m \times n \) matrix. Let \( r = \text{rank}(M) \). Then

\[
\begin{align*}
\dim \mathcal{R}(M) &= r \\
\dim \mathcal{R}(M^T) &= r \\
\dim \mathcal{N}(M) &= n - r \\
\dim \mathcal{N}(M^T) &= m - r
\end{align*}
\]

Also

\[
\begin{align*}
\mathcal{N}(M) &= \mathcal{R}(M^T)^\perp \\
\mathcal{N}(M^T) &= \mathcal{R}(M)^\perp
\end{align*}
\]

where \( \perp \) signifies the annihilator of the set.

Only the second to last line needs a proof. Here it is: consider first \( x \in \mathcal{N}(M) \) and show that \( x \) belongs to the annihilator of \( \mathcal{R}(M^T) \), in other words, that \( x^T(M^T y) = 0 \) for all \( y \in F^m \), which is obvious since \( x^T(M^T y) = (Mx)^T y \). Conversely, taking \( x \in \mathcal{R}(M^T)^\perp \), this means that \( 0 = x^T(M^T y) = (Mx)^T y \) for all \( y \in F^m \), which means that \( Mx = 0 \), hence \( x \in \mathcal{N}(M) \). \( \square \)