Recall: \( \{e^{inx}\}_{n \in \mathbb{Z}} \) form an orthogonal basis

in \( L^2(-\pi, \pi) \). That is, any \( f \in L^2(-\pi, \pi) \) can be uniquely represented as a Fourier series

\[
f \sim \sum_{n=\infty}^{\infty} \hat{f}_n e^{inx}
\]

and the sequence \( \{\hat{f}_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \), that is, \( \sum_{n=\infty}^{\infty} |\hat{f}_n|^2 < \infty \),

**Sin-Cos series**: when \( f \) has real values for \( x \in \mathbb{R} \) we may prefer to write a series with real terms

\[
[f \text{ minimally have } \hat{f}(n) \equiv \hat{f}_n]
\]

so \( s_f(x) = \hat{f}(0) + \sum_{n=1}^\infty (\hat{f}_n e^{inx} + \hat{f}_n^* e^{-inx}) \in \mathbb{R} \iff \hat{f}_n = \overline{f}_n
\]

So if \( \hat{f}(n) = A_n + i B_n \) and \( \hat{f}(-n) = A_n - i B_n \) then

\[
s_f(x) = \hat{f}(0) + \sum_{n=1}^\infty 2 \Re\left( e^{inx} \hat{f}_n \right) = \hat{f}(0) + \sum_{n=1}^\infty 2(A_n \cos nx - B_n \sin nx)
\]

Denote \( a_n = 2A_n, \ b_n = -2B_n \).

Then \( a_n = \frac{\hat{f}(n) + \hat{f}(-n)}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( e^{inx} + e^{-inx} \right) f(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(nx) f(x) \, dx \)

and \( b_n = \frac{1}{2} (\hat{f}(n) - \hat{f}(-n)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( e^{inx} - e^{-inx} \right) f(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2i \sin(nx) f(x) \, dx \)

while \( \hat{f}_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(0x) f(x) \, dx = \frac{1}{2} a_0 \)

for convenience!
So: if \( f(x) \) is real-valued, we can either write 
\[
f \sim \sum_{n = -\infty}^{\infty} \hat{f}_n e^{inx}
\]
where 
\[
\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) \, dx
\]
and for \( f(x) \) real then \( \hat{f}(x) = \overline{\hat{f}(x)} \) so \( \hat{f}_{-n} = \overline{\hat{f}_n} \)

or we can write
\[
f \sim \frac{a_0}{2} + \sum_{n = 1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
where
\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x) \, dx, \quad n = 0, 1, 2, \ldots
\]
\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x) \, dx, \quad n = 1, 2, \ldots
\]
and
\[
a_n = 2 \Re \Re \hat{f}_n
\]
\[
b_n = -2 \Im \Im \hat{f}_n
\]

The set of functions
\[
\{ 1, \cos nx, \sin nx, \cos 2nx, \sin 2nx, \ldots \}
\]
form an orthogonal basis in \( L^2(-\pi, \pi) \).

\*
\*

In other intervals, \( f(x) \in L^2(a, b) \to \mathbb{R} \)
\[
f \sim \sum_{n} \hat{f}_n e^{i \frac{2\pi}{b-a} x} \quad \Rightarrow \quad \hat{f}_n = \frac{1}{b-a} \int_{a}^{b} e^{-i \frac{2\pi}{b-a} x} f(x) \, dx
\]
\[
f \sim \frac{a_0}{2} + \sum a_n \cos \left( \frac{2\pi}{b-a} x \right) + b_n \sin \left( \frac{2\pi}{b-a} x \right)
\]
Using either representations, we know that

If \( f \in L^2(-\pi, \pi) \) then its Fourier series converges in mean square sense, that is,

\[
S_N(f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[ a_n \cos(nx) + b_n \sin(nx) \right]
\]

we have \( \lim_{N \to \infty} \| f - S_N(f) \|_2 = 0 \)

that is,

\[
\lim_{N \to \infty} \| f - S_N(f) \|_2^2 = \lim_{N \to \infty} \int_{-\pi}^{\pi} \left( f(x) - S_N(f)(x) \right)^2 \, dx = 0
\]

**Question**: When does the Fourier series converge for each particular \( x \)?

1. For which \( x \) does \( \lim_{N \to \infty} S_N(f)(x) \) exist?
2. Moreover, for which \( x \) does this limit equal \( f(x) \)?

The Dirichlet kernels is very useful in finding the answers. We will see they help with \( S_N(f) \).
The Dirichlet kernels are
\[ D_N(x) = \sum_{k=-N}^{N} e^{ikx} \]

On, in terms of this function,
\[ D_N(x) = 1 + \sum_{k=1}^{N} (e^{ikx} + e^{-ikx}) = 1 + 2 \sum_{k=1}^{n} \cos(kx) \]

Other normalizations used:
\[ \frac{1}{2} \left( 1 + 2 \sum_{k=1}^{n} \cos(kx) \right) \]

or
\[ \frac{1}{2\pi} D_N(x) := \sigma_N(x) \quad \text{(we will work with this)} \]

Simple formula:
\[ D_N(x) = \sum_{k=-N}^{N} e^{ikx} = e^{-iNx} \sum_{k=0}^{2N} e^{i(k+N)x} \]

(Recall \( 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a} \))

\[ = e^{-iNx} \frac{1 - e^{i(2N+1)x}}{1 - e^{ix}} = \frac{e^{-iNx} - e^{i(N+1)x}}{1 - e^{ix}} \]

\[ = \frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}} \]

\[ \sigma_N(x) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}} \]
Properties of \( f_N(x) \):

- are \( 2\pi \)-periodic (since all \( e^{inx} \) are)
- are even functions (since \( e^{ix} \) is even)
- \( f_N(x) = 0 \) \( \forall x \) \( (N+\frac{1}{2})x = k\pi \) \( (k \in \mathbb{Z}) \)

and choosing the zeroes in \([-\pi, \pi]\) they turn out to be

\[ \frac{2k\pi}{2N+1} \] \( \forall k = 0, \pm 1, \ldots, \pm N \) \( (2N+1 \text{ in all}) \)

- since between any zeroes there must be at least one point of max or min, and it turns out there is exactly one, there are \( N \) max and \( N \) min
Given $f \in L^2(-\pi, \pi) \to \mathbb{R}$

$$f \sim \sum_{n=\infty}^{\infty} \hat{f}_n e^{inx}$$

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}_n e^{inx} = \frac{a_0}{2} \sum_{n=1}^{N} \left[ a_n \cos(nx) + b_n \sin(nx) \right]$$

$$\downarrow$$

Symmetric sum, to get a real expression.

Note: $S_N(f)$ is a trigonometric polynomial, and its Fourier coefficients are

$$\left( \hat{S}_N \right)_n = \begin{cases} \hat{f}_n & \text{if } -N \leq n \leq N \\ 0 & \text{if } |n| > N \end{cases}$$

So the sequence $\left( \hat{S}_N \right)_n \in \mathbb{L}^2(\mathbb{Z})$ and it is a product

$$\left( \hat{S}_N \right) \cdot \hat{g}_N$$

where $(\hat{g}_N)_n = \begin{cases} 1 & \text{if } -N \leq n \leq N \\ 0 & \text{if } |n| > N \end{cases}$

in the sense that $\left( \hat{S}_N \right)_n \cdot \hat{g}_N$

Now, $\hat{g}_N$ does not depend on $f$! It is a universal cut-off sequence.

Going back to functions, it means that

$$S_N(x) = "\text{some operator}" \text{ with } f \text{ and this universal function } g_N$$

where $g_N(x) = \sum_{n=-N}^{N} (\hat{g}_N)_n e^{inx} = \sum_{n=-N}^{N} e^{inx} = D_N(x) \| |$

This operator is called **convolution**.
Indeed,
\[
\int_{-\pi}^{\pi} \phi(\theta) : D_N(x-\theta) \, d\theta = \int_{-\pi}^{\pi} \left( \sum_{k=-\infty}^{\infty} \hat{\phi}_k e^{ik\theta} \right) \left( \sum_{l=-N}^{N} e^{i(l-x)\theta} \right) \, d\theta \\
= \sum_{k=-\infty}^{\infty} \sum_{l=-N}^{N} \hat{\phi}_k e^{ik\theta} \int_{-\pi}^{\pi} e^{i(l-x)\theta} \, d\theta \\
= 2\pi \sum_{k=-N}^{N} \hat{\phi}_k e^{ik\theta} = 2\pi S_N(\hat{\phi})(x)
\]

So
\[
S_N(\hat{\phi})(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\theta) D_N(x-\theta) \, d\theta \\
= \int_{-\pi}^{\pi} \phi(\theta) \cdot S_N(x-\theta) \, d\theta \\
\Rightarrow S_N(\hat{\phi}) = \hat{\phi} \times S_N
\]

(the reason to work with \(S_N\) rather than \(D_N\);

on Stein's book, the convolution is normalized, be careful)

All this trouble is due to the fact that we work with \(e^{inx}\) with \(\|e^{inx}\|_2 = \sqrt{2\pi}\).

If we worked instead with the orthonormal basis
\[
\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_n,
\]
then we do not need to worry about normalization and the Fourier transform is
a unitary operator. But it has been entrenched otherwise, and habits (good or bad) are hard to break.\]
Definition: Given two functions \( f, g \) on an interval \([a, b]\), then their convolution is
\[
(f * g)(x) = \int_a^b f(x-s)g(s)\,ds.
\]

Properties of convolution

Assuming \( f, g \) are nice functions (say, continuous on \([a, b]\)) then
\[
\begin{align*}
\&f * g = g * f \quad \rightarrow \text{commutative} \\
\&f * (g + h) = f * g + f * h \quad \rightarrow \text{bilinear} \\
\&f * (cg) = cf * g \quad (\text{for } c \text{ constant}) \\
\&f * (g * h) = (f * g) * h \quad \rightarrow \text{associative}
\end{align*}
\]

Exercise: Prove these.

And most importantly, if \( f, g \) are \( 2\pi \)-periodic,
\[
\hat{f * g} = \hat{f} \cdot \hat{g}.
\]

(The \( 2\pi \) is the price to pay since \( \| e^{inx} \|_2 = \frac{\sqrt{2\pi}}{n} \)).

Proof
\[
(f * g)_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} (f * g)(x)\,dx
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \left( \int_{-\pi}^{\pi} f(x-s)g(s)\,ds \right)\,dx
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-inx} f(x-s)g(s)\,ds\,dx
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-inx} f(x-s)g(s)\,ds\,dx
\]

\( x-s = y \)
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, e^{-i\omega \theta} \int_{-\pi}^{\pi} dy \, g(y) \, e^{-i y \theta} = \int_{-\pi}^{\pi} dy \, g(y) \, e^{-i y \theta} \quad \text{since } g \text{ is } 2\pi - \text{periodic}
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, e^{-i\omega \theta} \hat{f}(\theta) \int_{-\pi}^{\pi} dy \, g(y) \, e^{-i y \theta} = 2\pi \hat{f}_n \delta_n^*.
\]

**Corollary**

\[
S_N(f) = f * \delta_N
\]

\[
\sum_{n=-N}^{N} \hat{f}_n e^{inx} = \int_{-\pi}^{\pi} f(\theta) \, \delta_N(x - \theta) \, d\theta
\]

**Corollary to Corollary**

The linear operator \( P : L^2(-\pi, \pi) \rightarrow L^2(-\pi, \pi) \)

\[
Pf = f * \delta_N
\]

is the orthogonal projection onto

\[
\text{span} \{1, e^{ix}, \ldots, e^{ixN}\} = \text{span} \{1, \cos x, \sin x, \ldots, \cos Nx, \sin Nx\}
\]

Therefore \( f * \delta_N \) gives the best approximation of \( f \) (in least mean square sense) of \( f \) by trigonometric polynomials in \( \cos kx, \sin kx, k=1,2,\ldots, N \).