Recall: $\{e^{inx}\}_{n\in\mathbb{Z}}$ form an orthogonal basis for $L^2(-\pi,\pi)$. That is, any $f \in L^2(-\pi,\pi)$ can be uniquely represented as a Fourier series

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx}$$

and the sequence $\{\hat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, that is, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty$

$\boxed{\text{Sin-Cos series}}$:

when f has real values $f(x) \in \mathbb{R}$ we may prefere to write a series with real terms [For simplicity here $\hat{f}(n) \equiv \hat{f}_n$]

so $Sf(x) = \hat{f}(0) + \sum_{n=1}^{\infty} \left(e^{inx}\hat{f}(n) + e^{-inx}\hat{f}(-n)\right) \in \mathbb{R}$ if $\hat{f}_{-n} = \overline{\hat{f}_n}$

So if $\hat{f}(n) = A_n + iB_n$ and $\hat{f}(-n) = A_n - iB_n$ then

$$Sf(x) = \hat{f}(0) + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\left(e^{inx}\hat{f}(n)\right) = \hat{f}(0) + \sum_{n=1}^{\infty} 2\left(A_n \cos(nx) - B_n \sin(nx)\right)$$

$$= \hat{f}(0) + \sum_{n=1}^{\infty}\Big(\underbrace{2A_n}_{a_n} \cos nx \underbrace{-\,2B_n}_{b_n} \sin nx\Big)$$

Denote $a_n = 2A_n$, $b_n = -2B_n$.

Then $\underline{a_n} = \hat{f}(n) + \hat{f}(-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{inx} + e^{-inx}\right) f(x)\,dx = \underline{\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx) f(x)\,dx}$

and $\underline{b_n} = \frac{-1}{i}\left(\hat{f}(n) - \hat{f}(-n)\right) = \frac{-1}{2\pi i}\int_{-\pi}^{\pi}\left(e^{-inx} - e^{inx}\right) f(x)\,dx = \underline{\frac{1}{2\pi i}\int_{-\pi}^{\pi}(2i\,\sin(nx)) f(x)\,dx}$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\, f(x)\,dx$$

while $\underline{\hat{f}_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}\,\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(0\cdot x) f(x)\,dx = \underline{\frac{1}{2}a_0}$ for convenience!

So: if $f(x)$ is real-valued, $f:[-\pi,\pi)\to\mathbb{R}$ we can either write $f \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$

where $\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$

and for $f_{(x)}$ real then $f(x) = \overline{f(x)}$ so $\hat{f}_{-n} = \overline{\hat{f}_n}$

or we can write

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx) f(x)\, dx$, $n=0,1,2\ldots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx) f(x) dx$, $n = 1,2\ldots$

and $a_n = 2\,\mathrm{Re}\,\hat{f}_n$

$b_n = -2\,\mathrm{Im}\,\hat{f}_n$

The set of functions $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$ form an orthogonal basis for $L^2(-\pi,\pi)$

* * *

For other intervals: $f \in L^2(a,b) \to \mathbb{R}$

$f \sim \sum \hat{f}_n e^{in\frac{2\pi x}{b-a}}$, $\hat{f}_n = \frac{1}{b-a}\int_a^b e^{in\frac{2\pi x}{b-a}} f(x) dx$

$f \sim \frac{a_0}{2} + \sum a_n \cos(n\frac{2\pi x}{b-a}) + b_n \sin(n\frac{2\pi x}{b-a})$

Using either representations, we know that if $f \in L^2(-\pi,\pi)$ then its Fourier series converges to f in mean square sense, that is,
if we take truncates of the Fourier series

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)]$$

we have $\lim_{N\to\infty} \|f - S_N(f)\|_{L^2} = 0$

that is,

$$\lim_{N\to\infty} \|f - S_N(f)\|_{L^2}^2 = \lim_{N\to\infty} \int_{-\pi}^{\pi} [f(x) - S_N(f)(x)]^2 dx = 0$$

Question: when does the Fourier serie converge for each particular x?

- for which x does $\lim_{N\to\infty} S_N(f)(x)$ exist?
- moreover, for which x does this limit equal $f(x)$?

The Dirichlet kernels is very useful for finding the answers. We will see they help with $S_N(f)$.

<u>Def</u> The Dirichlet kernels are

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx}$$

Or, in terms of trig functions

$$D_N(x) = 1 + \sum_{k=1}^{N} \left(e^{ikx} + e^{-ikx}\right) = 1 + 2\sum_{k=1}^{n} \cos(kx)$$

Other normalizations used:

$$\frac{1}{2}\left(1 + 2\sum_{k=1}^{n} \cos(kx)\right)$$

or $\quad \frac{1}{2\pi} D_N(x) \quad := \delta_N(x) \quad$ (we will work with this)

Simpler formula:

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx} = e^{-iNx} \sum_{k=0}^{2N} e^{i(k+N)x}$$

(Recall $1 + a + a^2 + \dots + a^n = \frac{1 - a^{n+1}}{1-a}$)

$$= e^{-iNx} \frac{1 - e^{i(2N+1)x}}{1 - e^{ix}} = \frac{e^{-iNx} - e^{i(N+1)x}}{1 - e^{ix}}$$

$$= \frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

$$\boxed{\delta_N(x) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}}$$

Properties of $\sigma_N(x)$:

- are 2π-periodic (since all e^{inx} are)
- are even functions (since cos is even)
- $\sigma_N(x) = 0$ for $(N+\frac{1}{2})x = k\pi$ $(k \in \mathbb{Z})$

  and choosing the zeros in $[-\pi, \pi]$ they turn out to be $\frac{2k\pi}{2N+1}$ for $k = 0, \pm 1, \ldots \pm N$ ($2N+1$ in all)

- since between any 2 zeros there must be at least one point of max or min, and it turns out there is exactly one, there are N max and N min

Given $f \in L^2(-\pi,\pi) \to \mathbb{R}$

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$$

$$S_N(f)_{(x)} = \sum_{n=-N}^{N} \hat{f}_n e^{inx} = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

symmetric sum,
to get a real expression

<u>Note</u>: $\underbrace{S_N(f)}_{=h}$ is a <u>trigonometric polynomial</u>, and its Fourier coefficients are $\widehat{(S_N f)}_n = \begin{cases} \hat{f}_n & \text{if } -N \le n \le N \\ 0 & \text{if } |n| > N \end{cases}$

So the sequence $\{\hat{h}_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and it is a product $\widehat{S_N}(f) = \hat{f} \cdot \hat{g}_N$ where $(\hat{g}_N)_n = \begin{cases} 1 & \text{if } -N \le n \le N \\ 0 & \text{if } |n| > N \end{cases}$

in the sense that $\widehat{(S_N f)}_n = \hat{f}_n \hat{g}_N$

<u>Now</u>: $\hat{g}_N$ does not depend on f! It is a universal cut-off sequence.

Going back to functions, it means that

$S_N(x) =$ "some operation" with f and this universal function g_N

where $g_N(x) = \sum_{n=-\infty}^{\infty} (\hat{g}_N)_n e^{inx} = \sum_{n=-N}^{N} e^{iNx} = D_N(x)$!!!

This operation is called <u>convolution</u>.

Indeed:

$$\int_{-\pi}^{\pi} f(s)\, D_N(x-s)\, ds =$$

$$\int_{-\pi}^{\pi}\Big(\sum_{k=-\infty}^{\infty} \hat{f}_k e^{iks}\Big)\Big(\sum_{\ell=-N}^{N} e^{i\ell(x-s)}\Big)\, ds$$

$$= \sum_{k=-\infty}^{\infty}\sum_{\ell=-N}^{N} \hat{f}_k e^{i\ell x} \underbrace{\int_{-\pi}^{\pi} e^{i(k-\ell)s}\, ds}_{\substack{=0 \text{ for } k\neq \ell \\ =2\pi \text{ for } k=\ell}}$$

$$= 2\pi \sum_{k=-N}^{N} \hat{f}_k e^{ikx} = 2\pi\, S_N(f)(x)$$

So $$S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\, D_N(x-s)\, ds$$

$$= \int_{-\pi}^{\pi} f(s)\cdot \delta_N(x-s)\, ds \implies \boxed{S_N(f) = f * \delta_N}$$

(the reason to work with δ_N rather than D_N; or, in Stein's book, the convolution is normalized, be careful)

[All this trouble is due to the fact that we work with e^{inx} with $\|e^{inx}\|_{L^2} = \sqrt{2\pi}$.

If we worked instead with the orthonormal basis $\left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\}_n$ then we do not need to worry about normalization and the Fourier transform is a unitary operator. But it has been entrenched otherwise, and habits (good or bad) are hard to break...]

<u>Definition</u>: given 2 functions f, g on an interval $[a,b]$ then their <u>convolution</u> is

$$(f*g)(x) = \int_a^b f(x-s)\,g(s)\,ds.$$

<u>Properties of convolution</u>

Assuming f, g are nice functions (say, continuous on $[a,b]$) then

$f*g = g*f$ → commutative

$$\left.\begin{aligned} f*(g+h) &= f*g + f*h \\ f*(cg) &= c(f*g) \quad (\text{for } c = \text{constant}) \end{aligned}\right\} \text{bilinear}$$

$f*(g*h) = (f*g)*h$ → associative

<u>Exercise</u>: prove these

And, most importantly, $\boxed{\text{Theorem}}$ for f, g 2π-periodic,

$$\widehat{f*g} = 2\pi\, \hat{f}\,\hat{g}$$

(the 2π is the price to pay since $\| e^{inx} \|_{L^2} = \sqrt{2\pi} \neq 1$)

<u>Proof</u>

$$(\widehat{f*g})_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} (f*g)(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, e^{-inx} \int_{-\pi}^{\pi} f(s)\,g(x-s)\,ds$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} ds\; f(s)\,g(x-s)\,e^{-in(x-s)}\,e^{-ins}\,ds$$

$x - s = y$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} ds\, e^{-ins} f(s) \underbrace{\int_{x-\pi}^{x+\pi} dy\, g(y)\, e^{-iny}}$$

$$= \int_{-\pi}^{\pi} dy\; g(y)\, e^{-iny}$$ since g is 2π-periodic

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} ds\, e^{-ins} f(s)\, ds \int_{-\pi}^{\pi} dy\; g(y)\, e^{-iny}$$

$$= 2\pi\, \hat{f}_n\, \hat{g}_n \quad . \quad \square$$

<u>Corollary</u>

$$\boxed{S_N(f) = f * \delta_N}$$

or
$$\sum_{n=-N}^{N} \hat{f}_n e^{inx} = \int_{-\pi}^{\pi} f(s)\, \delta_N(x-s)\, ds$$

<u>Corollary to Corollary</u>

The linear operator $P: L^2(-\pi,\pi) \longrightarrow L^2(-\pi,\pi)$

$$Pf = f * \delta_N$$

is the orthogonal projection onto $Sp\{1, e^{\pm ix}, \ldots e^{\pm iNx}\}\left(= Sp\{1, \cos x, \sin x, \ldots \cos Nx, \sin Nx\}\right)$

Therefore $f*\delta_N$ gives the best approximation of f (in least mean square sense) of f by trigonometric polynomials in $\cos kx, \sin kx$, $k=0,1,2 \ldots N$.