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1. Setting the stage

For a long time mathematicians, physicists, engineers worked, particularly in problems in differential equations, by differentiating functions which are not differentiable, integrating against "functions" which are zero everywhere except at one point, and obtaining a nonzero integral, integrating functions whose integral is not defined, and, in the end, obtaining valid results!

The ideas and techniques have been gathered and set on a rigorous mathematical ground by Sobolev (1930's), and, separately, by Laurent Schwartz (1940's). The main idea in working with generalized functions is that these are only needed and defined in integrals, that is, they are linear functionals.

Distributions are continuous linear functionals on spaces of *test functions.*

1.1. **Test functions.** These are defined on a domain of interest for a particular problem: on $\mathbb{R}$, or $\mathbb{R}^2$, $\mathbb{R}^n$ or a subset $(a,b) \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^n$) and are very, very, nice functions, having all the properties one may wish for in tackling a problem at hand. Namely:
- they are infinitely many times differentiable (we say for short that they are class C^∞), and
- they vanish, together with all their derivatives, at the boundary of the domain (so that when we integrate by parts the boundary terms vanish).

Here are the most popular spaces of test functions.

$\mathcal{S}(\mathbb{R})$, the *rapidly decaying functions* (the Schwartz space) contains functions which decay, towards $\pm\infty$, more rapidly than any power, and so do all their derivatives. For example, $\phi(x) = e^{-x^2} \in \mathcal{S}(\mathbb{R})$.

It can be shown that the Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is one-to-one and onto. This requires a proof, but here is the intuitive reason: f has decay $\leftrightarrow \mathcal{F}f$ is smooth, and f is smooth $\leftrightarrow \mathcal{F}f$ has decay, and functions in $\mathcal{S}$ are both smooth and have decay. $\mathcal{S}$ is Fourier heaven!

$C_0^\infty(\Omega)$, the *functions with compact support* on $\Omega \subset \mathbb{R}^n$ are functions which are non-zero only on a bounded set, and they are identically zero before the boundary of Ω. For example,

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \in C_0^\infty(\mathbb{R})$$

and in fact $\phi(x) \in C_0^\infty(-a,a)$ for any $a > 1$.

Note: the *spaces of test functions are linear subspaces of* L^2 (and they are dense in L^2).

A sequence of test functions ϕ_n is said to *converge* to a test function ϕ (in the same space) if it converges in the strongest sense you may imagine (not only point-wise, but also all the derivatives $\phi_n^{(k)}(x) \to \phi^{(k)}(x)$, and, moreover, they converge uniformly on any compact subset... this ensures that one may pass to the limit freely all the operations needed: summation, differentiation, integration without any worry).

1.2. **Distributions.** Given a space of test functions, *distributions are continuous linear functionals*, that is, if u is a distribution

$$\text{we denote } u(\phi) := (\phi, u) \quad \text{for any test function } \phi$$

and

$$(c\phi + d\psi, u) = c(\phi, u) + d(\psi, u) \quad \text{for test functions } \phi, \psi \text{ and constants } c, d$$

$$\text{whenever } \phi_n \to \phi \text{ as test functions, then } (\phi_n, u) \to (\phi, u)$$

1.3. **Examples of distributions.** In what follows we assume our test functions are $C_0^\infty(\mathbb{R})$. For other spaces of test functions minimal extra care may be needed.

1.3.1. *Function type distributions.* Any function which is not too wild (i.e. it is integrable on compact sets) can be regarded as a distribution: say, $f(x) = 1$, or $f(x) = \sin x$, or $f(x) = e^{ix}$, or $f(x) = H(x)$, defines a distribution by the formula

$$(\phi, f) = \int_{-\infty}^{\infty} f(x)\phi(x)\, dx \tag{1}$$

The integral converges since any test function $\phi(x)$ vanishes outside some interval $[-M, M]$.[1] The functional defined ny (1) is clearly linear; a bit of mathematical argumentation would also show continuity (which we do not pursue here).

Note the underlying idea: (1) is, essentially, the inner product of L^2. We want to use it for more general functions, like the examples above, which are not in L^2; we do that, but the price we pay is that we can only do pairing with very rapidly decaying functions.

In the same spirit: when working with complex-valued functions, to use the full force of Hilbert space theory we need to define *distributions as conjugate-linear functionals*, that is, $u(\phi) := \langle \phi, u \rangle$ satisfies

$$\langle c\phi + d\psi, u \rangle = \overline{c}\langle \phi, u \rangle + \overline{d}\langle \psi, u \rangle$$

The constructions and definitions for distributions defined as linear functionals can be easily transcribed for distributions defined as conjugate-linear functionals (and we will use the inner product notation in this latter case).

[1] Of course, if the test functions are $\mathcal{S}$, then we should require f not to increase faster than some power.

1.4. **Dirac's delta function.** δ is defined as

$$(\phi, \delta) = \phi(0) \tag{2}$$

and shifted delta, $\delta(x-a)$is defined as

$$(\phi, \delta(x-a)) = \phi(a) \tag{3}$$

Note that (2),(3) are usually written, in the spirit of (1),

$$(\phi, \delta) = \int_{-\infty}^{\infty} \phi(x)\delta(x)\, dx = \phi(0)$$

and

$$(\phi, \delta(x-a)) = \int_{-\infty}^{\infty} \phi(x)\delta(x-a)\, dx = \int_{-\infty}^{\infty} \phi(t+a)\delta(t)\, dt = \phi(a)$$

though this are not a bonafide integrals (but work like integrals).

2. Operations with distributions

2.1. **Linear combinations.** Any linear combination of distributions is a distribution (why?).

2.2. **Multiplication by C^∞ functions.** We can multiply distributions u with C^∞ functions $f(x)$ and we obtain another distribution fu, defined as

$$(\phi, fu) = (f\phi, u) \quad \text{for any test function } \phi$$

(the definition is, of course, inspired by what happens for function type distributions). Note that the definition makes sense since if $\phi \in C_0^\infty(\mathbb{R})$, and $f \in C^\infty(\mathbb{R})$, hence $\phi f \in C_0^\infty(\mathbb{R})$.

Note: When we work with distributions over the test functions $\mathcal{S}$ (these care called *tempered distributions*) we also need to require that the function f increases no faster than some power at $\pm\infty$, to ensure that if $\phi \in \mathcal{S}$ then also $f\phi \in \mathcal{S}$.

2.3. **Any distribution is differentiable.** If our distribution is a nice, differentiable function $f(x)$, what should its derivative be? As a distribution, let us see how it acts on test functions:

$$(\phi, f') = \text{ [since } f' \text{ is a function] } = \int_{-\infty}^{\infty} \phi(x) f'(x) dx$$

and integrating by parts

$$= \phi(x)f(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi'(x)f(x)dx = \text{ [since } \phi(\pm\infty) = 0] \ = -(\phi', f)$$

This justifies

Definition 1. *The derivative u' of a distribution u is defined by the rule*

$$(\phi, u') = -(\phi', u) \quad \textit{for any test function } \phi$$