4.4. **Autonomous equations.** If the coefficients of $L = L(\frac{d}{dx})$ do not depend on x, the equation $Lu = f(x)$ is called autonomous. For example, (6), (9) are autonomous equations.

Autonomous equations are translation-invariant, and we want to take advantage of that. More precisely, if $u(x)$ solves $Lu = 0$ then $u(x-t)$ solves the same (homogeneous) equation for any t. And is $u(x)$ solves $Lu = f(x)$ then also $u(x-t)$ solves $L[u(x-t)] = f(x-t)$ for any t. So instead of searching for a Green's function $G(x,t)$ by solving $LG = \delta(x-t)$ with boundary conditions, we could first solve for $LF = \delta$, then add to $F(x-t)$ the general solution of the homogeneous equation, and then impose the boundary conditions, the result being the Green's function. The procedure is illustrated below. But first, a definition.

Definition 5. *Let $L = L(\frac{d}{dx})$ be an autonomous differential operator.*

A solution $F(x)$ of $LF = \delta$ is called a fundamental solution *of the differential operator L.*

Then $u_{part}(x) = (f * F)(x) = \int f(t)F(x-t)dt$ is *a particular solution* of $Lu = f$. The general solution of $Lu = f$ is $u_{\text{part}} + u_{\text{homog}}$ where u_{homog} is the general solution of $Lu = 0$.

4.4.1. *A simple example.* Consider the autonomous problem (6) (which, of course, is a particular case of (7), for $a(x) \equiv \alpha$). Its fundamental solution is found as in §4.3.1 (only calculating for $t = 0$): we find a solution of

$$\frac{dF}{dx} + \alpha F = \delta(x) \tag{15}$$

(no conditions imposed). Multiplying be the integrating factor $e^{\alpha x}$ and integrating we obtain

$$F(x) = e^{-\alpha x}\left[H(x) + C\right]$$

Any particular value for C gives a fundamental solution of L. For example, for $C = -\frac{1}{2}$ we obtain the symmetric fundamental solution $F(x) = \frac{1}{2}\,\text{sign}(x)\, e^{-\alpha x}$.

To find the Green's function of L with condition $u(x_0) = 0$, for $x \geq x_0$ we look for it in the form $F(x-t) + C(t)e^{-\alpha x}$ (note that the free constant in the solution of the homogeneous equation is allowed to depend on t). We determine $C(t)$ by imposing the boundary condition: $G(x_0,t) = e^{-\alpha(x_0-t)}H(x-t) - C(t)e^{-\alpha(x_0-t)} = 0$ and solving for $C(t)$ we get

$$G(x,t) = e^{-\alpha(x-t)}\left(H(x-t) - H(x_0-t)\right)$$

The solution of the initial value problem is then given by $u(x) = \int_{x_0}^{\infty} G(x,t)f(t)\,dt$ (which for $x > x_0$ equals the familiar $u(x) = e^{-\alpha x}\int_{x_0}^{x} e^{\alpha t} f(t)\,dt$).

Alternatively, the solution can be found as follows. A particular solution of $\frac{du}{dx} + \alpha u = f$ is then (for, say, $C = 0$)

$$u_{\text{part}}(x) = (f * F)(x) = \int_{-\infty}^{\infty} f(t)e^{-\alpha(x-t)}H(x-t)\,dt$$