Examples of Fourier integrals

1. If $f(x)$ is an even function then $(\mathcal{F}f)(\xi)$ is real.

Indeed, $(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx$

$$= \frac{1}{\sqrt{2\pi}} \Big(\underset{\substack{\uparrow \\ \text{set here } x=-t}}{\int_{-\infty}^{0}} + \int_0^{\infty} \Big) = \frac{1}{\sqrt{2\pi}} \Big(\int_0^{\infty} e^{+it\xi} \underbrace{f(-t)}_{=f(t)}\,dt + \int_0^{\infty} e^{-ix\xi} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\, 2 \int_0^{\infty} \cos(\xi t) f(t)\,dt \in \mathbb{R}$$

2. If $f(x)$ is odd then $(\mathcal{F}f)(\xi)$ is purely imaginary

Indeed, as above $(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} (-2i) \int_0^{\infty} \sin(\xi t) f(t)\,dt \in i\mathbb{R}$

3. Rectangle function $f(x) = \begin{cases} 1 & \text{if } x \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$ (is even)

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}}\, 2 \int_0^{a} \cos(\xi t)\,dt = \frac{1}{\sqrt{2\pi}}\, 2\, \frac{\sin(\xi a)}{\xi}$$ sinc function!

4. <u>Note</u>: $(\mathcal{F}f)(-t) = (\mathcal{F}^{-1}f)(t)$

5. $\mathcal{F}$ of an exponentially decreasing function $f(x) = \begin{cases} e^{-\beta x}, & x > 0 \\ 0, & x < 0 \end{cases}$ $(\beta > 0)$

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ix\xi - \beta x}\,dx = \frac{1}{\sqrt{2\pi}} \frac{1}{i\xi + \beta}$$ decreases like $\frac{1}{\xi}$!

6. $\mathcal{F}(f') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f'(x)\,dx \underset{\text{by parts}}{=} \frac{1}{\sqrt{2\pi}}\left(\underbrace{e^{-ix\xi} f(x)\Big|_{-\infty}^{\infty}}_{=0} + i\xi \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\right)$

$$\boxed{\mathcal{F}(f') = i\xi\,\mathcal{F}(f)}$$

"$\mathcal{F}$" diagonalizes the symmetric operator $-i\frac{d}{dx}$

as $\mathcal{F}(-i\frac{df}{dx}) = \xi\,\mathcal{F}(f)$

<u>Exercise</u> check that $-i\frac{d}{dx}$ is essentially self-adjoint on $L^2(\mathbb{R})$

7. $$\boxed{\mathcal{F}(e^{-x^2/2}) = e^{-\xi^2/2}}$$

<u>Recall</u> $\int e^{-x^2/2}\,dx$ cannot be expressed in terms of known functions, but we can calculate $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$

<u>Here is how</u>: we calculate $I = \int_0^\infty e^{-x^2/2}\,dx$

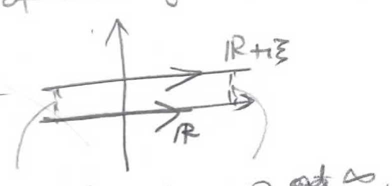
by calculating $I^2 = \left(\int_0^\infty e^{-x^2/2}\,dx\right)^2 = \int_0^\infty\int_0^\infty e^{-x^2/2-y^2/2}\,dx\,dy$

go to polar coordinates $= \int_0^\infty dr \int_0^{\pi/2} d\theta\; e^{-r^2/2}\, r$

$= \frac{\pi}{2}\left(-e^{-r^2/2}\right)\Big|_0^\infty = \frac{\pi}{2}$ hence $\boxed{\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}}$

<u>Now</u> $\mathcal{F}(e^{-x^2/2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi - \frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac12(x+i\xi)^2 - \xi^2/2}\,dx = e^{\xi^2/2}$

where we used path deformation for integrals of analytic functions in $\mathbb{C}$: $\int_{-\infty}^{\infty} e^{-\frac12(x+i\xi)^2}\,dx = \int_{\mathbb{R}+i\xi} e^{-\frac12 t^2}\,dt = \int_{\mathbb{R}} e^{-t^2/2}\,dt$



integrals $\to 0$ at ∞.

8. $\mathcal{F}$ of shift

$$\mathcal{F}(f(x+c)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x+c)\,dx \underset{\substack{\uparrow \\ x+c=t}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\xi} e^{-ic\xi} f(t)\,dt$$

$$\boxed{\mathcal{F}(f(x+c)) = e^{-ic\xi}\,\mathcal{F}(f)}$$

9. $\mathcal{F}$ of dialation : $\mathcal{F}(f(ax)) = ?$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(ax)\,dx \underset{ax=t}{=} \frac{1}{a}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\xi/a} f(t)\,dt$$

$$\boxed{\mathcal{F}(f(ax))(\xi) = \frac{1}{a}\,\mathcal{F}(f)\left(\frac{\xi}{a}\right)}$$

<u>Note:</u> stretching the graph of f $\Rightarrow$ we squeeze the graph of $\mathcal{F}f$

squeezing the graph of f $\Rightarrow$ we stretch $\mathcal{F}f$

"f cannot be localized both in space and in frequency"

10. $(\mathcal{F}f)(\mathcal{F}g) = ?$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx \int_{-\infty}^{\infty} e^{-iy\xi} g(y)\,dy = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(x+y)\xi} f(x) g(y)\,dy \quad x+y=t$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{-it\xi} \int_{-\infty}^{\infty} dx\, f(x)\, g(t-x) = \frac{1}{\sqrt{2\pi}}\,\mathcal{F}(f*g)$$

where $\boxed{(f*g)(t) = \int_{-\infty}^{\infty} dx\, f(x)\, g(t-x)}$

$$\boxed{\mathcal{F}(f)\,\mathcal{F}(g) = \frac{1}{\sqrt{2\pi}}\,\mathcal{F}(f*g)}$$

11. Recall that $\boxed{\mathcal{F}^{-1}(f) \equiv \mathcal{F}(\check{f})}$

where $\check{f}(x) = f(-x)$

and, of course, $\mathcal{F}(f) = \mathcal{F}^{-1}(\check{f})$

$\mathcal{F}^{-1}(\check{f}) = \mathcal{F}(f)$

Conversely, $\mathcal{F}(fg) = ?$

In the formula $(\mathcal{F}f)(\mathcal{F}g) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(f*g)$ take $\mathcal{F}^{-1}$

and denote $\mathcal{F}f = F$, $\mathcal{F}g = G$

$FG = \frac{1}{\sqrt{2\pi}}\mathcal{F}(\mathcal{F}^{-1}(F)*\mathcal{F}^{-1}(G))$

so $\mathcal{F}^{-1}(FG) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}(F)*\mathcal{F}^{-1}(G)$

or $\mathcal{F}(\check{F}\check{G}) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(\check{F})*\mathcal{F}(\check{G})$ for all $\check{F}, \check{G} \in L^2(\mathbb{R})$ for which $*$ makes sense

hence $\boxed{\mathcal{F}(fg) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(f)*\mathcal{F}(g)}$

Using $\mathcal{F}$ to solve differential equations:

12. Example: Solve $u'' - u = f$. Apply $\mathcal{F}$ to both sides (searching for $u \in L^2(\mathbb{R})$ and assuming $f \in L^2(\mathbb{R})$)

Using $\mathcal{F}(f') = i\xi\mathcal{F}(f)$ we get

$(-\xi^2 - 1)\mathcal{F}u = \mathcal{F}(f)$ hence $\mathcal{F}u = -\frac{\mathcal{F}(f)}{\xi^2+1}$

so $u = -\mathcal{F}^{-1}\left(\frac{1}{\xi^2+1}\cdot\mathcal{F}(f)\right) = -\mathcal{F}^{-1}\left(\frac{1}{\xi^2+1}\right)*f\ \sqrt{2\pi}$

$\frac{1}{\xi^2+1} = \frac{1}{(\xi+i)(\xi-i)} = \frac{1}{2i}\left(\frac{1}{\xi-i} - \frac{1}{\xi+i}\right)$ and we may even calculate $\mathcal{F}^{-1}\left(\frac{1}{\xi^2+1}\right)$. <u>Exercise</u> How?