When do Fourier series converge (point-wise)?

Recall that \( f \) is called differentiable at \( x \) if
\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists; then } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

and \( f'(x) \) = slope of the tangent line to the graph of \( f \) at the point \((x, f(x))\).

\[
\tan \theta = f'(x)
\]

**Def.** \( f \) is said to have a right-hand derivative if
\[
\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \text{ exists; } f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}
\]

slope of the tangent line to the part of the graph to the right of \( x \).

Similarly
**Def.** \( f \) has a left-hand derivative if
\[
\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \text{ exists; } f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}
\]
Example of a function which has both left and right derivatives, but it is not differentiable:

Ex. \( f(x) = |x| \) has right and left derivatives at \( x = 0 \)

\[ f'_+(0) = 1 \quad \text{and} \quad f'_-(0) = -1 \]

**Theorem**
Suppose \( f \) is piecewise continuous on \((-\pi, \pi)\) and periodic on \( \mathbb{R} \) with period \( 2\pi \).

Then at each point \( x \) where \( f'_+(x), f'_-(x) \) exist

\[
\lim_{N \to \infty} S_n(f)(x) = \frac{f(x+) + f(x-)}{2}
\]

In particular, if \( f \) is continuous at \( x \) then

\[
\lim_{N \to \infty} S_n(f)(x) = f(x)
\]

**Remark**
Be careful about \( x = \pm \pi \)

the function

\( f \) is periodic, period \( 2\pi \)

but the points \( x = -\pi \) and \( x = \pi \) are points of discontinuity (for its periodic continuation from \([-\pi, \pi)\) to \( \mathbb{R} \) and

\[
\lim_{N \to \infty} S_n(f)(\pm \pi) = \frac{f(\pi-) + f(-\pi+)}{2} = \frac{b}{2}.
\]
Remark if piecewise continuous \( f \in L^1(0, \pi) \)

so \( f \) has a Fourier series.

Note

- There are many theorems of this type: if \( f \) is a
  nice function in this and that sense, then its Fourier
  series converges to that.

  But this theorem is quite useful in applications.

- If \( f \) does not satisfy the hypotheses of the theorem,
  it does not mean that the Fourier series does not
  converge at \( x \)! (The theorem gives sufficient conditions
  for convergence, they are not necessary.)

For the proof, we first need two lemmas.

The first lemma is a special case of

**Riemann-Lebesgue Lemma:**

If \( f \) is integrable then

\[
\lim_{n \to \infty} \int_a^b f(x) \sin(nx) \, dx = 0
\]

\[
\lim_{n \to \infty} \int_a^b f(x) \cos(nx) \, dx = 0
\]

or, together,

\[
\lim_{n \to \infty} \int_a^b f(x) e^{inx} \, dx = 0
\]

**Read:** if \( f \in L^1(0, b) \) then \( f_n \to f \) pointwise as \( n \to \infty \).

We will discuss later the duality: the more regular \( f \) is,
the more rapidly \( f_n \to 0 \) as \( n \to \infty \).
It was assumed that $\int_{-\pi}^{\pi} |f(x)| < \infty$, that is, $f \in L^1(-\pi, \pi)$ (and not in $L^2$!) This is a more stringent condition.

For example, $\frac{1}{x^2/3} \in L^1(-\pi, \pi)$ but $\not\in L^2(-\pi, \pi)$.

In general, note for bounded intervals

$$f \in L^2(a, b) \implies f \in L^1(a, b)$$

(because by Cauchy-Schwarz

$$\|f\|_{L^1} = \int_a^b |f(x)| \, dx = \langle f, 1 \rangle \leq \|f\|_{L^2} \|1\|_{L^2} \leq \frac{1}{\sqrt{\infty}} = b-a$$

So: $L^1(a, b) \subset L^2(a, b)$
Lemma 1

If \( G(x) \) is a function piecewise continuous on \((0, \pi)\)
then \( \lim_{N \to \infty} \int_0^{\pi} G(x) \sin (N+\frac{1}{2})x \, dx = 0 \) \((\text{for } N \in \mathbb{Z}_+)\)

Proof

\[
\int_0^{\pi} G(x) \sin (N+\frac{1}{2})x \, dx = \int_0^{\pi} G(x) \sin^2 \frac{x}{2} \sin Nx \, dx + \int_0^{\pi} G(x) \sin \frac{x}{2} \cos Nx \, dx
\]

where \( a_N, b_N \) are coeff in the Fourier series on \( G \cos \), respectively \( G \sin \)

Since the Fourier coef \( e^{-iNz} \to a_N \to 0, b_N \to 0 \)

Recall \( \sigma_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi} \frac{\sin (N+\frac{1}{2})x}{\sin \frac{x}{2}} \)

Note that \( \int_0^{\pi} \sigma_N(x) \, dx = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_0^{\pi} e^{inx} \, dx = \frac{1}{2\pi} \cdot \pi = \frac{1}{2} \)

Lemma 2

Suppose \( g(x) \) is piecewise cont and \( g'(0) \) exist
Then \( \lim_{N \to \infty} \int_0^{\pi} g(x) \sigma_N(x) \, dx = \frac{1}{2} g(0+) \)

Proof

Write \( g(x) = [g(x) - g(0^+)] + g(0^+) \)
Then \( \int_0^{\pi} \sigma_N(x) g(0^+) \, dx = \frac{1}{2} g(0^+) \), while
\[
\int_0^N \left[ g(x) - g(0+) \right] \delta_N(x) = \int_0^N \frac{g(x) - g(0^+)}{\sin x} \min \left( N + \frac{1}{2}, x \right) \, dx
\]

\[
= g(x), \text{ piecewise continuous}
\]

Since
\[
\lim_{x \to 0^+} \frac{g(x) - g(0^+)}{\sin x} = \lim_{x \to 0^+} \frac{g(x) - g(0^+)}{x} = g'(0) = 2
\]

So by Lemma 1 has zero limit

\[\square\]

**Proof of the Theorem**

Recall:

**Def.** A function \( f \) is called **piecewise continuous** on \((a, b)\) if there are a finite number of intervals, divided at

\[a = a_0 < a_1 < a_2 < a_3 < \ldots < a_p = b\]

so that \( f \) is continuous on each interval \((a, a_1), (a_1, a_2), \ldots, (a_p, b)\)

and \( f \) has lateral limits at each \( a_0, a_1, \ldots, a_p \):

Thus exist (and are finite) \( \lim_{x \to a_k^-} f(x) = f(a_k^-) \)

and \( \lim_{x \to a_k^+} f(x) = f(a_k^+) \)

(\( f \) need not be defined at \( a_0, a_1, \ldots, a_p \))

**Ex.**

\[a \quad a_0 \quad a_1 \quad a_2 \quad b\]

piecewise continuous function
The point is that on each subinterval $f: (a_k, a_{k+1}) \rightarrow \mathbb{R}$ is continuous and it could be extended to a continuous function on $[a_k, a_{k+1}]$, hence $f$ has a max and a min on $[a_k, a_{k+1}]$ and it is Riemann integrable:

$$\int_{a_k}^{a_{k+1}} f(x) \, dx = \sum_{k=0}^{p} \int_{a_k}^{a_{k+1}} f(x) \, dx$$ well-defined.

So $f \in L^2$ also, and $f$ has a Fourier series

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Consider the partial sums

$$(S_N f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

We have

$$(S_N f)(x) = \delta_N * f = \int_{-\pi}^{\pi} f(s-x) \delta_N(s) \, ds = \int_{-\pi}^{\pi} f(s-x) \delta_N(s) \, ds - \int_{-\pi}^{\pi} f(s-x) \delta_N(s) \, ds$$

and using Lemma 2:

We have

$$\lim_{N \to \infty} (S_N f)(x) = \frac{1}{2} f(x+) + \frac{1}{2} f(x-)$$
**Example**

The Fourier series of

\[ f(x) = x + \pi \text{ for } x \in (-\pi, \pi) \]

then continued \(2\pi\)-periodic.

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \cdot \frac{2\pi \cdot 2\pi}{2} = 2\pi \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2\sin(n\pi)}{n} = 0 \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{\sin(n\pi) - \cos(n\pi) \cdot n\pi}{n^2 \pi} = \frac{(-1)^n n\pi}{n^2} \]

**Fourier Series**

\[ 2\pi + \sum_{n=1}^{\infty} (-1)^n \frac{n\pi}{n^2} \sin(nx) = \sqrt{\int f(x)^2 \, dx} \text{ for } x \in (-\pi, \pi) \]

\[ 0 + \frac{2\pi}{\pi} = 2\pi \]
Corollary: If $f$ is $2\pi$ periodic on $\mathbb{R}$, continuous, and has left and right derivatives at all points $x$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{for all } x$$