Gibbs Phenomenon

$$2\pi S_n f(x) = \int_{-\pi}^{\pi} f(s)\, D_N(x-s)\, ds$$

$D_N(t) = \dfrac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$ has a huge max at $x=0$, equal to $2N+1$
has zeros at $\dfrac{2k\pi}{2N+1}$

N large $\rightarrow$ dense zeros $\rightarrow$ rapid oscillating

The main contribution in the integral is thus collected around $x=s$. If f is continuous at x then $f(s) \approx f(x)$ on this interval so the integral $\rightarrow f(x)\cdot 2\pi$ Fine!

Now take an example with f discontinuous at, say, $x=0$

$$H(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$$

$$2\pi S_n H(x) = \int_0^{\pi} D_N(x-s)\, ds. \quad \text{for } x>0. \text{ Take also } x \text{ small}$$

$x - s = \sigma$

$$= \int_{x-\pi}^{x} D_N(\sigma)\, d\sigma \quad \text{. where } x \text{ small, } x>0$$

and main contribution is around $\sigma = 0$ so

$$\approx \int_{-x}^{x} D_N(\sigma)\, d\sigma \underset{\uparrow \text{even}}{=} 2\int_0^x D_N(\sigma)\, d\sigma = 2\int_0^x \frac{\sin(N+\frac{1}{2})\sigma}{\sin\sigma/2}\, d\sigma$$

$$\text{since } x \text{ is small} \approx 4\int_0^x \frac{\sin(N+\frac{1}{2})\sigma}{\sigma}\, d\sigma \underset{(N+\frac{1}{2})\sigma = \tau}{=} 4\int_0^{(N+\frac{1}{2})x} \frac{\sin\tau}{\tau}\, d\tau \equiv F(x)$$

$F(0)=0$, F increases for small x, so its max is at the first zero of its derivative: $\sin(N+\frac{1}{2})x = 0$ first time for $x = \dfrac{\pi}{N+\frac{1}{2}}$

and its value is $4\displaystyle\int_0^{\pi} \frac{\sin\tau}{\tau}\, d\tau$

$$\text{so } S_N H(x) \approx \frac{2}{\pi}\int_0^{\pi} \frac{\sin\tau}{\tau}\, d\tau \approx 1.1 \qquad \underline{\underline{\text{Overshoot!}}}$$

The more derivatives f has, the faster $\hat{f}_n$ decay:

Let $f(x)$ be 2π-periodic,
continuous, piecewise differentiable.
Then $\hat{f}_n$ decay (at least as fast as) $\frac{1}{n^2}$:

$$\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$$ Let x_0 where $f'_\pm(x_0)$ exist.

Then $$\hat{f}_n = \frac{1}{2\pi}\left(\int_{-\pi}^{x_0} e^{-inx} f(x)\,dx + \int_{x_0}^{\pi} e^{-inx} f(x)\,dx\right)$$

integrate by parts

$$= \frac{1}{2\pi}\left[\frac{-1}{in} e^{-inx} f(x)\Big|_{-\pi}^{x_0} + \frac{1}{in}\int_{-\pi}^{x_0} e^{-inx} f'(x)\,dx\right.$$
$$\left. + \frac{-1}{in} e^{-inx} f(x)\Big|_{x_0}^{\pi} + \frac{1}{in}\int_{-\pi}^{x_0} e^{-inx} f'(x)\,dx\right]$$

since f' is integrable on $[-\pi,\pi]$ and since $f(-\pi)=f(\pi)$

$$= \frac{1}{2\pi}\,\frac{-1}{in}\,e^{-inx_0}\underbrace{\left[f(x_0+)-f(x_0-)\right]}_{\text{if } \neq 0 \text{ then } \hat{f}_n \sim \frac{e^{-inx_0}\, i}{2\pi n}} + \frac{1}{2\pi}\,\frac{1}{in}\int_{-\pi}^{\pi} e^{-inx} f'(x)\,dx$$

But, if f continuous at x_0

$$= \frac{1}{2\pi i n}\int_{-\pi}^{\pi} e^{-inx} f'(x)\,dx.$$

Continue: If f' is 2π periodic, and

If f' jumps at x_0, then $\hat{f}_n$ behaves like $\frac{1}{n^2}$

but if f'' exists piecewise and f'' periodic then $\hat{f}_n$ behaves like $\frac{1}{n^3}$

Etc.

The more derivatives, the faster the decay of $\hat{f}_n$.