Gibbs Phenomenon

\[ 2\pi S_n f(x) = \pi \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \sin \frac{nx}{2} \]

has a huge max at \( x=0 \), \( \approx 2\pi \) at \( x=\pm \frac{\pi}{n} \)

For \( N \) large \( \Rightarrow \) dense maxes \( \Rightarrow \) rapid oscillations

The main contribution in the integral is thus collected around \( x=0 \).
If \( f \) is continuous at \( x \) then \( f(0) \approx f(x) \) in this interval so the integral \( \approx f(x) \times 2\pi \) Envelope.

Now take an example with \( f \) discontinuous at \( x=0 \), \( f(0) = 0 \)

\[ f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \]

For \( x > 0 \) Take also \( x \) small

\[ 2\pi S_n f(x) = \int_{0}^{\pi} D_n(x,0) \, dx \quad \text{where } x \text{ small}, x > 0 \]

and main contribution is around \( x = 0 \)

\[ \approx \int_{-\pi}^{\pi} D_n(0,0) \, d\sigma \quad \text{even} \]

\[ = 2 \int_{0}^{\pi} D_n(0,0) \, d\sigma = 2 \int_{0}^{\pi} \frac{\sin \left( \frac{N+1}{2} \right) \sigma}{\sin \frac{\sigma}{2}} \, d\sigma \]

\( \text{since } x \text{ is small} \quad \approx 4 \int_{0}^{\pi} \frac{\sin \left( \frac{N+1}{2} \right) \sigma}{\sigma} \, d\sigma = 4 \int_{0}^{\pi} \frac{\sin \frac{\sigma}{2}}{\sigma} \, d\sigma \quad \text{for } \sin \left( \frac{N+1}{2} \right) \sigma = 0 \]

\( \text{if } f(0) = 0 \), finesses \( f \) small \( x \), so its max is at the first zero of its derivative \( \sin \left( \frac{N+1}{2} \right) x = 0 \text{ first time } \sin x = \frac{\pi}{N+\frac{1}{2}} \)

and its value is \( \approx 4 \int_{0}^{\pi} \frac{\sin \frac{\sigma}{2}}{\sigma} \, d\sigma \)

\[ \approx 2 \pi \int_{0}^{\pi} \frac{\sin \frac{\sigma}{2}}{\sigma} \, d\sigma \approx 1.1 \quad \text{Overshoot!} \]
The more derivatives $f$ has, the faster $f_n$ decay.

Let $f(x)$ be $2\pi$-periodic, continuous, piecewise differentiable.

Then $f_n$ decay (at least as fast as) $\frac{1}{n^2}$:

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) \, dx$$

Let $x_0$ where $f'(x_0)$ exist.

Then

$$f_n = \frac{1}{2\pi} \left( \int_{-\pi}^{x_0} e^{-inx} f(x) \, dx + \int_{x_0}^{\pi} e^{-inx} f(x) \, dx \right)$$

Integrate by parts

$$= \frac{1}{2\pi} \left[ \frac{1}{in} e^{-inx} f(x) \right]_{-\pi}^{x_0} + \frac{1}{in} \int_{-\pi}^{x_0} e^{-inx} f'(x) \, dx$$

$$+ \frac{1}{in} e^{-inx} f(x) \int_{x_0}^{\pi} + \frac{1}{in} \int_{x_0}^{\pi} e^{-inx} f'(x) \, dx \right]$$

Since $f'$ is integrable on $[-\pi, \pi]$ and $\lim_{x \to \pm \pi} f(x) = f(x)$

$$= \frac{1}{2\pi} \frac{1}{in} e^{-inx} \left[ f(x_0) - f(x_{0-}) \right] + \frac{1}{2\pi} \frac{1}{in} \int_{-\pi}^{\pi} e^{-inx} f'(x) \, dx$$

If $f$ is continuous at $x_0$

$$= \frac{1}{2\pi in} \int_{-\pi}^{\pi} e^{-inx} f'(x) \, dx.$$  \text{Conclude: If } f' \text{ is } 2\pi \text{ periodic, and}

If $f'$ jumps at $x_0$, then $f_n$ behaves like $\frac{1}{n^2}$

but if $f''$ exists piecewise then $f_n$ behaves like $\frac{1}{n^3}$

Etc.

The more derivatives, the faster the decay of $f_n$.  