3. Hilbert Spaces

In a Hilbert space we can do linear algebra (since it is a vector space), geometry (since we have lengths and angles) and calculus (since it is complete). And they are all combined when we write series expansions.

Recall:

Definition 23. *A Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.*

Recall the two fundamental examples: the space of sequences ℓ^2, and the space of square integrable function L^2.

3.1. When does a norm come from an inner product?

In every Hilbert space the **parallelogram identity** holds: for any $f, g \in H$

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2) \tag{6}$$

(in a parallelogram the sum of the squares of the sides equals the sum of the squares of the diagonals).

Relation (6) is proved by a direct calculation (expanding the norms in terms of the inner products and collecting the terms).

The remarkable fact is that, if the parallelogram identity holds in a Banach space, then its norm actually comes from an inner product, and can recover the inner product only in terms of the norm by
the polarization identity:
for complex spaces

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right)$$

and for real spaces

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right)$$

3.2. The inner product is continuous.

This means that we can take limits inside the inner product:

Theorem 24. *If $f_n \in H$, with $f_n \to f$ then $\langle f_n, g\rangle \to \langle f, g\rangle$.*
If also $g_n \in H$, with $g_n \to g$ then $\langle f_n, g_n\rangle \to \langle f, g\rangle$.
In particular, $\|f_n\| \to \|f\|$.

Indeed: $|\langle f_n, g\rangle - \langle f, g\rangle| = |\langle f_n - f, g\rangle| \le$ (by Cauchy-Schwarts) $\|f_n - f\|\,\|g\| \to 0$. (Recall: $f_n \to f$ means that $\|f_n - f\| \to 0$.)

For prove the second statement we use a trick which is quite standard in similar circumstances:

$$\begin{aligned} |\langle f_n, g_n - \rangle \langle f, g\rangle| &= |\langle f_n, g_n\rangle - \langle f_n, g\rangle + \langle f_n, g\rangle - \langle f, g\rangle| \\ \le |\langle f_n, g_n\rangle - \langle f_n, g\rangle| &+ |\langle f_n, g\rangle - \langle f, g\rangle| = |\langle f_n, g_n - g\rangle| + |\langle f_n - f, g\rangle| \\ &\le \|\|f_n\|\, \|g_n - g\| + \|f_n - f\|\, \|g\| \end{aligned}$$

were we used the triangle inequality (for numbers), then Cauchy-Schwartz. Now, $\|f_n - f\| \to 0$, hence the last tens goes to zero, and so does the first term, since $\|g_n - g\| \to 0$, and since $\|f_n\|$ is bounded ($f_n \to f$, hence $\|f_n\| \to \|f\|$ - why?, therefore $\|f_n\| \le \|f\| + (\text{say})1$ if n is large enough). □

3.3. **Orthonormal bases.**

Consider a Hilbert space H.

Just like in linear algebra we define:

Definition 25. *If $\langle f, g\rangle = 0$ then $f, g \in H$ are called orthogonal ($f \perp g$).*

and

Definition 26. *A set $B \subset H$ is called orthonormal if all $f, g \in B$ are orthogonal ($f \perp g$) and unitary ($\|f\| = 1$).*

Note that as in linear algebra, an orthonormal set is a linearly independent set (why?).

Recall: the *Pythagorean identity*:

$$if\ \ f \perp g \quad then\ \ \|f + g\|^2 = \|f\|^2 + \|g\|^2$$

And departing from linear algebra:

Definition 27. *A set $B \subset H$ is called an* **orthonormal basis** *for H if it is an orthonormal set and it is complete in the sense that the span of B is* dense *in H: $\overline{Sp(B)} = H$.*

Note that in the above B is not a basis in the sense of linear algebra (unless H is finite-dimensional). But like in linear algebra:

Theorem 28. *Any Hilbert space admits an orthonormal basis; furthermore, any two orthonormal bases of the same space have the same cardinality, called the Hilbert dimension of the space.*

From here on we will only consider Hilbert spaces which admit a (finite or) countable orthonormal basis.

These are the Hilbert spaces encountered in mechanics and electricity.

It can be proved that this condition is equivalent to the existence of a countable set S dense in H. This condition is often easier to check, and the property is called "H is separable". Application: $L^2[a, b]$ is separable (why?) therefore $L^2[a, b]$ has a countable orthonormal basis. Many physical problems are solved by finding special orthonormal basis of $L^2[a, b]$!

[For your amusement: it is quite easy to construct a Hilbert space with an uncountable basis, e.g. just like we took $\ell^2(\mathbb{Z}_+)$ we could take $\ell^2(\mathbb{R})$.]

It is easy and instructive to prove Theorem 28 for separable Hilbert spaces: pick any basis (it is countable) and use a Gram-Schmidt process!

The following theorem shows that many of the properties of inner product vector spaces which are finite dimensional (think $\mathbb{R}^n$ or $\mathbb{C}^n$ with the standard basis) are very similar for Hilbert spaces; the main difference is that in infinite dimensions instead of sums we have series - which means sums followed by limits, see for example the infinite linear combination (7).

Theorem 29. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space with a countable basis.*

Let $u_1, \ldots, u_n, ..$ be an orthonormal basis.

The following hold.

(i) Let $c_1, \ldots, c_n, ..$ be scalars so that $(c_1, \ldots, c_n, ..) \in \ell^2$. Then the series

$$f = \sum_{n=1}^{\infty} c_n u_n \tag{7}$$

converges and its sum $f \in \mathcal{H}$.

Moreover

$$\|f\|^2 = \sum_{n=1}^{\infty} |c_n|^2$$

(ii) Conversely, every $f \in \mathcal{H}$ has an expansion (7). The scalars c_n satisfty $c_n = \langle u_n, f\rangle$ and are called **generalized Fourier coefficients** *of f.*

Therefore any $f \in \mathcal{H}$ can be written as a **generalized Fourier series***:*

$$f = \sum_{n=1}^{\infty} \langle u_n, f\rangle\, u_n \tag{8}$$

and **Parseval's identity** *holds:*

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle u_n, f\rangle|^2 \tag{9}$$

As a consequence, **Bessel's inequality** *holds:*

$$\|f\|^2 \geq \sum_{n\in J} |f_n|^2 \qquad \text{for any } J \subset \mathbb{Z}_+ \tag{10}$$

(iii) If $f, g \in \mathcal{H}$ then

$$\langle f, g\rangle = \sum_{n=1}^{\infty} \langle f, u_n\rangle\, \langle u_n, g\rangle \tag{11}$$

Main steps of the proof of Theorem 29.

(i) The partial sums $s_N = \sum_{n=1}^N c_n u_n$ form a Cauchy sequence since (say $M > N$)

$$\|s_N - s_M\|^2 = \|\sum_{n=N+1}^{M} c_n u_n\|^2 = \sum_{n=N+1}^{M} |c_n|^2 \to 0 \text{ as } M, N \to \infty$$

since $\{c_n\}_n \in \ell^2$. Since $\mathcal{H}$ is complete, then s_N has a limit.

(ii) Denote $s_N = \sum_{n=1}^N \langle u_n, f\rangle\, u_n$ and show that $\lim_{N\to\infty} s_N = f$.

Note first that $s_N \perp f - s_N$ (an easy calculation), hence, by the Pythagorean Theorem, $\|f\|^2 = \|s\|^2 + \|f - s_N\|^2$ which implies $\|f\|^2 \geq \|s\|^2$. Since $\|s\|^2 = \sum_{n=1}^N |\langle u_n, f\rangle|^2$ (short calculation, just like in finite dimensions) it means that $\sum_{n=1}^\infty |\langle u_n, f\rangle|^2 < \infty$ hence the sequence $\{\langle u_n, f\rangle\}_n \in \ell^2$.

Then by (i), $\sum_{n=1}^\infty \langle u_n, f\rangle\, u_n = f_1 \in \mathcal{H}$. We only need to show that $f_1 = f$.

Note that $f - f_1 \perp$ all u_k (the finite dimensions intuition works, but check!). Therefore $f - f_1 \perp Sp(u_1, u_2, \ldots)$ which implies $f - f_1 \perp \overline{Sp(u_1, u_2, \ldots)}$ (why?) but this closure is $\mathcal{H}$, hence $f - f_1 = 0$. $\Box$.

Note that (separable) Hilbert spaces are essentially ℓ^2, since given an orthonormal basis $u_1, u_2, \ldots$, the elements $f \in H$ can be identified with the sequence of their generalized Fourier coefficients $(c_1, c_2, c_3, \ldots)$ which, by (9), belongs to ℓ^2.