4.8.2. *Solution of the problem* (33),(34) *using an integral operator.* Rewrite (35), (36) as $(L-\lambda)y=0$. We have non-zero solutions y (eigenfunctions of L) only when $Ker(L-\lambda)\neq\{0\}$.

Let us consider the totally opposite case, and look at complex numbers z for which $L-z$ is invertible, and with $(L-z)^{-1}$ bounded. Such numbers $z\in\mathbb{C}$ are said to form the *resolvent set* of L, $\rho(L)$; all the other numbers forms the *spectrum* of L, $\sigma(L)$. In general, the spectrum contains, besides eigenvalues, also other numbers.

Let us invert $L-z$; this means that for any f we solve $(L-z)y=f$ giving $y=(L-z)^{-1}f$. The operator $(L-z)^{-1}$ is called *the resolvent* of L, and the set of all complex numbers for which $L-z$ is invertible (and with bounded inverse) is called *the resolvent set* of L. But for Sturm-Liouville problems like the example (35), (36), it turms out that the spectrum consists only of eigenvalues, and the eigenfunctions form a basis of the Hilbert space. Indeed, let us find the resolvent set of our example.

We solve the differential equation

$$y''+zy=-f \tag{37}$$

with the boundary conditions

$$y(0)=0,\ y(\pi)=0 \tag{38}$$

Recall that the general solution of (37) is given by

$$C_1y_1+C_2y_2-y_1\int\frac{y_2}{W}f+y_2\int\frac{y_1}{W}f$$

where y_1,y_2 are two linearly independent solutions of the homogeneous equation $y''+zy=0$ and $W=W[y_1,y_2]=y_1'y_2-y_1y_2'$ is their Wronskian.

We need to distinguish the cases $z=0$ and $z\neq 0$.

I. If $z=0$, then $y_1=1,y_2=x$ and we can easily solve the problem (37), (38).

II. If $z\neq 0$, then $y_1=\exp(kx)$ and $y_2=\exp(-kx)$ where $k=\sqrt{-z}$ (note that k may be a complex number; in particular, if $z>0$ then $k=i\sqrt{z}$). Their Wronskian is $W=2k$, so the general solution of (37) has the form

$$C_1e^{kx}+C_2e^{-kx}-\frac{e^{kx}}{2k}\int_0^x e^{-ks}f(s)\,ds+\frac{e^{-kx}}{2k}\int_0^x e^{ks}f(s)\,ds$$

The boundary condition $y(0)=0$ implies $C_2=-C_1$ therefore

$$y(x)=C\left(e^{kx}-e^{-kx}\right)-\frac{e^{kx}}{2k}\int_0^x e^{-ks}f(s)\,ds+\frac{e^{-kx}}{2k}\int_0^x e^{ks}f(s)\,ds$$

$$\equiv C\left(e^{kx}-e^{-kx}\right)+\int_0^\pi g(x,s)\,f(s)\,ds \tag{39}$$

where

$$g(x,s)=\frac{-1}{2k}\left(e^{kx-ks}-e^{-kx+ks}\right)H(x-s)$$

Imposing the boundary condition $y(\pi) = 0$ we obtain that C must satisfy

$$C\left(e^{k\pi} - e^{-k\pi}\right) + \int_0^\pi g(\pi, s)\, f(s)\, ds = 0$$

Solving for C and substituting in (39) we obtain that the solution of the problem (37), (38) is an integral operator (17):

$$y(x) = \int_0^\pi G(x, s)\, f(s)\, ds \;=\; (L - z)^{-1} f$$

where $G(x, s)$ is the Green function of the problem

$$G(x, s) = g(x, s) \;-\; \frac{e^{kx} - e^{-kx}}{e^{k\pi} - e^{-k\pi}}\, g(\pi, s) \tag{40}$$

Note that G is not defined if $e^{k\pi} - e^{-k\pi} = 0$ which means for $ik \in \mathbb{Z}$. Since $k = \sqrt{-z}$ ($z \neq 0$), this means that for $z = n^2$ ($n = 1, 2, \ldots$) the Green function is undefined (for these values the denominator of G vanishes): the resolvent $(L - z)^{-1}$ does not exist for $z = \lambda_n = n^2$.

(You may wish to note that $G(x, s)$ is continuous (the discontinuity of $H(x - s)$ at $s = x$ does not result in a discontinuity of $G(x, s)$ because $g(x, x) = 0$).

It is clear that if z is real then $(L - z)^{-1}$ is self-adjoint because L is self-adjoint.

The study of operators on Hilbert spaces is the topic of Functional Analysis. In one of its chapters it is proved that integral operators (17) (and other similar operators, called *compact operators*) which are selfadjoint are very much like selfadjoint matrices, in that they have real eigenvalues μ_n and the corresponding eigenfunctions u_n form an orthonormal basis for the Hilbert space. The infinite dimensionality of the Hilbert space implies that there are infinitely many eigenvalues: a countable set, which, moreover, tend to zero: $\mu_n \to 0$. (Zero may also be an eigenvalue.)

Let us see how the eigenvalues μ_n and eigenfunctions of the resolvent $(L - z)^{-1}$ are related to the eigenvalues λ_n and eigenfunctions y_n of L.

We have $Ly_n = \lambda_n y_n$ hence $(L - z)y_n = (\lambda_n - z)y_n$ for any number z. If $L - z$ is invertible (we saw that this is the case for $z \neq \lambda_k$ for all k) then $y_n = (\lambda_n - z)(L - z)^{-1} y_n$ so y_n is an eigenfunctions of the resolvent $(L - z)^{-1}$ corresponding to the eigenvalue $\mu_n = (\lambda_n - z)^{-1}$.

Note that $\lambda_n \to \infty$ (since $\mu_n \to 0$).

Note the following quite general facts:

◦ Remark the functional calculus aspect: if λ_n are the eigenvalues of L then $(\lambda_n - z)^{-1}$ are the eigenvalues of $(L - z)^{-1}$ and they correspond to the same eigenvectors.

◦ Note again that the eigenvalues of L appear as values of z for which the Green function has zero denominators.

4.9. **General second order self-adjoint problems.** Here is a more general situation than the example studied in §4.8.

General problem: Find the values of the constant λ for which the equation

$$\frac{1}{\rho(x)}\,\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)\,y\,+\,q(x)y+\lambda y=0 \tag{41}$$

has non-identically zero solutions for $x\in[a,b]$ satisfying the boundary conditions

$$y(a)=0,\;\;y(b)=0 \tag{42}$$

We shall see that (41) is the most general linear second order equation.

This is an eigenvalue problem for the differential operator

$$L=-\frac{1}{\rho(x)}\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)-q(x)$$

The operator L is selfadjoint in a weighted L^2 space: $L^2([0.L],\rho(x)dx)$ on a domain where the bounday values vanish. Indeed, it is easy to check that, if $f(a)=0$, $f(b)=0$ then

$$\int_a^b \overline{Lf(x)}\,g(x)\,\rho(x)dx=\int_a^b \overline{f(x)}\,Lg(x)\,\rho(x)dx-p\overline{f}'g\Big|_a^b$$

Since $f'(a)$, $f'(b)$ have arbitrary values, then L is formally self-adjoint if $p(a)g(a)=0, p(b)=g(b)=0$ for al $g\in D(L^*)$. If $p(a)\neq 0$ and $p(b)\neq 0$ then $D(L)=D(L^*)$ and L is self-adjoint.

Then, as in §4.8, it follows (from the theory of compact operators) that L has a sequence of eigenvalues and the corresponding eigenfunctions form an orthonormal basis for the weighted L^2.

We will look more closely to these operators in the next chapter, which studies the Sturm-Liouville problem.