4.8.2. Solution of the problem (33),(34) using an integral operator. Rewrite (35), (36) as \((L - \lambda)y = 0\). We have non-zero solutions \(y\) (eigenfunctions of \(L\)) only when \(\text{Ker}(L - \lambda) \neq \{0\}\).

Let us consider the totally opposite case, and look at complex numbers \(z\) for which \((L - z)\) is invertible, and with \((L - z)^{-1}\) bounded. Such numbers \(z \in \mathbb{C}\) are said to form the resolvent set of \(L, \rho(L)\); all the other numbers forms the spectrum of \(L, \sigma(L)\). In general, the spectrum contains, besides eigenvalues, also other numbers.

Let us invert \(L - z\); this means that for any \(f\) we solve \((L - z)f = y\) giving \(y = (L - z)^{-1}f\). The operator \((L - z)^{-1}\) is called the resolvent of \(L\), and the set of all complex numbers for which \(L - z\) is invertible (and with bounded inverse) is called the resolvent set of \(L\). But for Sturm-Liouville problems like the example (35), (36), it turns out that the spectrum consists only of eigenvalues, and the eigenfunctions form a basis of the Hilbert space. Indeed, let us find the resolvent set of our example.

We solve the differential equation

\[
y'' + z y = -f
\]

with the boundary conditions

\[
y(0) = 0, \ y(\pi) = 0
\]

Recall that the general solution of (37) is given by

\[
C_1y_1 + C_2y_2 - y_1 \int_0^x \frac{y_2}{W} f + y_2 \int_0^x \frac{y_1}{W} f
\]

where \(y_1, y_2\) are two linearly independent solutions of the homogeneous equation \(y'' + z y = 0\) and \(W = W[y_1, y_2] = y_1'y_2 - y_1y'_2\) is their Wronskian.

We need to distinguish the cases \(z = 0\) and \(z \neq 0\).

\(I\). If \(z = 0\), then \(y_1 = 1, y_2 = x\) and we can easily solve the problem (37), (38).

\(II\). If \(z \neq 0\), then \(y_1 = \exp(kx)\) and \(y_2 = \exp(-kx)\) where \(k = \sqrt{-z}\) (note that \(k\) may be a complex number; in particular, if \(z > 0\) then \(k = i\sqrt{z}\)). Their Wronskian is \(W = 2k\), so the general solution of (37) has the form

\[
C_1e^{kx} + C_2e^{-kx} - \frac{e^{kx}}{2k} \int_0^x e^{-ks} f(s) ds + \frac{e^{-kx}}{2k} \int_0^x e^{ks} f(s) ds
\]

The boundary condition \(y(0) = 0\) implies \(C_2 = -C_1\) therefore

\[
y(x) = C \left( e^{kx} - e^{-kx} \right) - \frac{e^{kx}}{2k} \int_0^x e^{-ks} f(s) ds + \frac{e^{-kx}}{2k} \int_0^x e^{ks} f(s) ds
\]

\[
y(x) = C \left( e^{kx} - e^{-kx} \right) + \int_0^\pi g(x, s) f(s) ds
\]

where

\[
g(x, s) = \frac{-1}{2k} \left( e^{kx - ks} - e^{-kx + ks} \right) H(x - s)
\]
Imposing the boundary condition \( y(\pi) = 0 \) we obtain that \( C \) must satisfy

\[
C \left( e^{k\pi} - e^{-k\pi} \right) + \int_0^\pi g(\pi, s) f(s) \, ds = 0
\]

Solving for \( C \) and substituting in (37), (38) we obtain that the solution of the problem (37), (38) is an integral operator (17):

\[
y(x) = \int_0^\pi G(x, s) f(s) \, ds = (L - z)^{-1} f
\]

where \( G(x, s) \) is the Green function of the problem

\[
G(x, s) = g(x, s) - \frac{e^{kx} - e^{-kx}}{e^k - e^{-k}} g(\pi, s)
\]

Note that \( G \) is not defined if \( e^{k\pi} - e^{-k\pi} = 0 \) which means for \( ik \in \mathbb{Z} \). Since \( k = \sqrt{-z} \) \((z \neq 0)\), this means that for \( z = n^2 \) \((n = 1, 2, \ldots)\) the Green function is undefined (for these values the denominator of \( G \) vanishes): the resolvent \((L - z)^{-1}\) does not exist for \( z = \lambda_n = n^2 \).

You may wish to note that \( G(x, s) \) is continuous (the discontinuity of \( H(x - s) \) at \( s = x \) does not result in a discontinuity of \( G(x, s) \) because \( g(x, x) = 0 \)).

It is clear that if \( z \) is real then \((L - z)^{-1}\) is self-adjoint because \( L \) is self-adjoint.

The study of operators on Hilbert spaces is the topic of Functional Analysis. In one of its chapters it is proved that integral operators (17) (and other similar operators, called compact operators) which are selfadjoint are very much like selfadjoint matrices, in that they have real eigenvalues \( \mu_n \) and the corresponding eigenfunctions \( u_n \) form an orthonormal basis for the Hilbert space. The infinite dimensionality of the Hilbert space implies that there are infinitely many eigenvalues: a countable set, which, moreover, tend to zero: \( \mu_n \to 0 \). (Zero may also be an eigenvalue.)

Let us see how the eigenvalues \( \mu_n \) and eigenfunctions of the resolvent \((L - z)^{-1}\) are related to the eigenvalues \( \lambda_n \) and eigenfunctions \( y_n \) of \( L \).

We have \( Ly_n = \lambda_n y_n \) hence \((L - z)y_n = (\lambda_n - z)y_n \) for any number \( z \). If \( L - z \) is invertible (we saw that this is the case for \( z \neq \lambda_k \) for all \( k \)) then \( y_n = (\lambda_n - z)(L - z)^{-1} y_n \) so \( y_n \) is an eigenfunctions of the resolvent \((L - z)^{-1}\) corresponding to the eigenvalue \( \mu_n = (\lambda_n - z)^{-1} \).

Note that \( \lambda_n \to \infty \) (since \( \mu_n \to 0 \)).

Note the following quite general facts:

- Remark the functional calculus aspect: if \( \lambda_n \) are the eigenvalues of \( L \) then \((\lambda_n - z)^{-1}\) are the eigenvalues of \((L - z)^{-1}\) and they correspond to the same eigenvectors.
- Note again that the eigenvalues of \( L \) appear as values of \( z \) for which the Green function has zero denominators.

4.9. **General second order self-adjoint problems.** Here is a more general situation than the example studied in §4.8.
General problem: Find the values of the constant $\lambda$ for which the equation

\begin{equation}
\frac{1}{\rho(x)} \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) y + q(x)y + \lambda y = 0
\end{equation}

has non-identically zero solutions for $x \in [a, b]$ satisfying the boundary conditions

\begin{equation}
y(a) = 0, \quad y(b) = 0
\end{equation}

We shall see that (41) is the most general linear second order equation. This is an eigenvalue problem for the differential operator

$$L = \frac{1}{\rho(x)} \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) - q(x)$$

The operator $L$ is selfadjoint in a weighted $L^2$ space: $L^2([0,L], \rho(x)dx)$ on a domain where the boundary values vanish. Indeed, it is easy to check that, if $f(a) = 0$, $f(b) = 0$ then

$$\int_a^b Lf(x)g(x)\rho(x)dx = \int_a^b f(x)Lg(x)\rho(x)dx - \left[ pf'g \right]_a^b$$

Since $f'(a), f'(b)$ have arbitrary values, then $L$ is formally self-adjoint if $p(a)g(a) = 0, p(b) = g(b) = 0$ for all $g \in D(L^*)$. If $p(a) \neq 0$ and $p(b) \neq 0$ then $D(L) = D(L^*)$ and $L$ is self-adjoint.

Then, as in §4.8, it follows (from the theory of compact operators) that $L$ has a sequence of eigenvalues and the corresponding eigenfunctions form an orthonormal basis for the weighted $L^2$.

We will look more closely to these operators in the next chapter, which studies the Sturm-Liouville problem.