Hand in the solutions to problems ~~1,2a,2e,3,4,7~~
3, 4, 8
next Friday

HOMEWORK 3

1. "Equivalent Widths"

Suppose we define for a square-integrable function $f(t)$ and its Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f(t)\, dt$$

the *equivalent width* as

$$\Delta_t = \left| \frac{\int_{-\infty}^{\infty} f(t)\, dt}{f(0)} \right| ,$$

and the *equivalent Fourier width* as

$$\Delta_\omega = \left| \frac{\int_{-\infty}^{\infty} \hat{f}(\omega)\, d\omega}{\hat{f}(0)} \right| .$$

a) Show that

$$\Delta_t \Delta_\omega = const.$$

is *independent* of the function f, and determine the value of this $const.$

b) Determine the equivalent width and the equivalent Fourier width for the unnormalized Gaussian

$$f(t) = e^{-t^2/2b^2}$$

and compare them with its full width and the full width of its Fourier as defined by their inflection points.

Reminder: The Fourier transform of a Gaussian can be obtained by completing the square in the exponent and using the Cauchy-Goursat Theorem, which you should know by now. See the attached handout.

2. "Periodic Function via Convolution"

Consider the periodic train of Dirac delta "functions"

$$f(x) = \sum_{n=-\infty}^{\infty} \delta(x - nc)$$

with real period $c \neq 0$.

(a) FIND and DESCRIBE its Fourier transform $\hat{f}(k)$. What happens to $\hat{f}$ if c gets doubled?

(b) Let $p(x+c) = p(x)$ be a periodic function.

Prove or disprove: $p(x)$ is the convolution (Eq.($\star\star$) in problem 2 of Homework set 2) of a periodic train of Dirac delta functions with a non-periodic function, say $g(x)$ in $L^2(-\infty,\infty)$. What is $g(x)$?

(c) Find the Fourier transform

$$\hat{p}(k) \equiv \mathcal{F}[p](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} p(x)\, dx$$

Compute the Fourier transform of the auto correlation function and thereby show that it equals the "spectral intensity" (a.k.a. power spectrum) of f whenever f is a real-valued function. This equality is known as the *Wiener-Khintchine formula.*

6. "Matched Filter"

Consider a linear time-invariant system. Assume its response to a driving force $f(t)$ can be written as

$$\int_{-\infty}^{\infty} g(T-t) f_0(t) dt \equiv h(T).$$

Here $g(T-t)$, the "unit impulse response' (a.k.a. "Green's function"), is a function which characterizes the system completely. The system is said to be *matched* to the particular forcing function f_0 if

$$g(T) = \overline{f_0}(-T).$$

(Here the bar means complex conjugate.) In that case the system response to a generic forcing function $f(t)$ is

$$\int_{-\infty}^{\infty} \overline{f_0}(t-T) f(t) dt \equiv h(T).$$

A system characterized by such a unit impulse response is called a *matched filter* because its design is matched to the particular signal $f_0(t)$. The response $h(T)$ is called the *cross-correlation* between f and f_0.

a) Compute the total energy

$$\int_{-\infty}^{\infty} |h(T)|^2 dT$$

of the cross-correlation $h(T)$ in terms of the Fourier amplitudes

$$\hat{f}_0(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f_0(t) dt$$

and

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f(t) dt.$$

b) Consider the family of forcing functions

$$\{f_0(t), f_1(t), \cdots, f_N(t)\}$$

and the corresponding family of normalized cross correlations (i.e. the corresponding responses of the system)

$$h_k(T) = \frac{\int_{-\infty}^{\infty} \overline{f_0}(t-T) f_k(t) dt}{\left[\int_{-\infty}^{\infty} |f_k(t)|^2 dt\right]^{1/2}} \quad k = 0, 1, \cdots, N$$

Show that (i) $h_0(T)$ is the peak intensity, i.e., that

$$|h_k(T)|^2 \leq |h_0(T)|^2 \quad k = 0, 1, \cdots$$

Problem 8

Given: $\mathcal{F}: L^2(-\infty,\infty) \longrightarrow L^2(-\infty,\infty)$

$$f \mapsto \mathcal{F}[f] = \hat{f}$$

Prove or disprove

$$\langle f, f\rangle = \langle \hat{f}, \hat{f}\rangle \ \forall f \in L^2 \Rightarrow \langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \ \forall f, g \ L^2$$

Here L^2 is a complex inner product space:

$$\langle f, f\rangle = \int_{-\infty}^{\infty} \bar{f}(t) f(t)\, dt$$