1. “Equivalent Widths”

Suppose we define for a square-integrable function \( f(t) \) and its Fourier transform
\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f(t) \, dt
\]
the equivalent width as
\[
\Delta_t = \left| \int_{-\infty}^{\infty} f(t) \, dt \right| / f(0)
\]
and the equivalent Fourier width as
\[
\Delta_\omega = \left| \int_{-\infty}^{\infty} \hat{f}(\omega) \, d\omega \right| / \hat{f}(0)
\]

a) Show that
\[
\Delta_t \Delta_\omega = \text{const.}
\]
is independent of the function \( f \), and determine the value of this \( \text{const.} \).

b) Determine the equivalent width and the equivalent Fourier width for the unnormalized Gaussian
\[
f(t) = e^{-t^2/2b^2}
\]
and compare them with its full width and the full width of its Fourier as defined by their inflection points.

Reminder: The Fourier transform of a Gaussian can be obtained by completing the square in the exponent and using the Cauchy-Goursat Theorem, which you should know by now. See the attached handout.

2. “Periodic Function via Convolution”

Consider the periodic train of Dirac delta “functions”
\[
f(x) = \sum_{n=-\infty}^{\infty} \delta(x - nc)
\]
with real period \( c \neq 0 \).

(a) FIND and DESCRIBE its Fourier transform \( \hat{f}(k) \). What happens to \( \hat{f} \) if \( c \) gets doubled?

(b) Let \( p(x + c) = p(x) \) be a periodic function.

Prove or disprove: \( p(x) \) is the convolution (Eq. (**)) in problem 2 of Homework set 2) of a periodic train of Dirac delta functions with a non-periodic function, say \( g(x) \) in \( L^2(-\infty, \infty) \). What is \( g(x) \)?

(c) Find the Fourier transform
\[
\hat{p}(k) = \mathcal{F}[p](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} p(x) \, dx
\]
Compute the Fourier transform of the auto correlation function and thereby show that it equals the “spectral intensity” (a.k.a. power spectrum) of $f$ whenever $f$ is a real-valued function. This equality is known as the Wiener-Khintchine formula.

6. “Matched Filter”

Consider a linear time-invariant system. Assume its response to a driving force $f(t)$ can be written as

$$\int_{-\infty}^{\infty} g(T-t)f_0(t)dt \equiv h(T).$$

Here $g(T-t)$, the “unit impulse response” (a.k.a. “Green’s function”), is a function which characterizes the system completely. The system is said to be matched to the particular forcing function $f_0$ if

$$g(T) = f_0^*(-T).$$

(Here the bar means complex conjugate.) In that case the system response to a generic forcing function $f(t)$ is

$$\int_{-\infty}^{\infty} f_0^*(t-T)f(t)dt \equiv h(T).$$

A system characterized by such a unit impulse response is called a matched filter because its design is matched to the particular signal $f_0(t)$. The response $h(T)$ is called the cross-correlation between $f$ and $f_0$.

a) Compute the total energy

$$\int_{-\infty}^{\infty} |h(T)|^2dT$$

of the cross-correlation $h(T)$ in terms of the Fourier amplitudes

$$\hat{f}_0(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f_0(t)dt$$

and

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt.$$  

b) Consider the family of forcing functions

$$\{f_0(t), f_1(t), \cdots, f_N(t)\}$$

and the corresponding family of normalized cross correlations (i.e. the corresponding responses of the system)

$$h_k(T) = \frac{\int_{-\infty}^{\infty} f_0^*(t-T)f_k(t)dt}{\left[ \int_{-\infty}^{\infty} |f_k(t)|^2dt \right]^{1/2}} \quad k = 0, 1, \cdots, N$$

Show that (i) $h_0(T)$ is the peak intensity, i.e., that

$$|h_k(T)|^2 \leq |h_0(T)|^2 \quad k = 0, 1, \cdots$$
Problem 2

Given: \( F: L^2(-\infty, \infty) \to L^2(-\infty, \infty) \)
\[ f \mapsto F[f] = f \]

Prove or disprove

\[ \langle \hat{f}, \hat{g} \rangle = \langle \hat{f}, \hat{f} \rangle \forall f \in L^2 \Rightarrow \langle \hat{f}, \hat{g} \rangle = \langle \hat{f}, \hat{g} \rangle \forall f, g \in L^2 \]

Here, \( L^2 \) is a complex inner product space:

\[ \langle f, f \rangle = \int_{-\infty}^{\infty} \overline{f(t)} f(t) \, dt \]