## 5. The Rayleigh's principle and the minimax principle for the eigenvalues of a self-adjoint matrix

Eigenvalues of self-adjoint matrices are easy to calculate. This section shows how this is done using a minimization, or maximization procedure.

### 5.1. The Rayleigh's quotient.

**Definition 49.** *Let $A = A^*$ be a self-adjoint matrix.* **The Rayleigh's quotient** *is the function*

$$R(\mathbf{x}) = \frac{\langle \mathbf{x}, A\mathbf{x}\rangle}{\|\mathbf{x}\|^2}, \quad \text{for } \mathbf{x} \neq \mathbf{0}$$

Note that

$$R(\mathbf{x}) = \langle \frac{\mathbf{x}}{\|\mathbf{x}\|}, A\frac{\mathbf{x}}{\|\mathbf{x}\|}\rangle = \langle \mathbf{u}, A\mathbf{u}\rangle \quad \text{where } \mathbf{u} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

so in fact, it suffices to define the Rayleigh's quotient on unit vectors.

The set of unit vectors in $\mathbb{R}^n$ (or in $\mathbb{C}^n$), is called the $n-1$ dimensional sphere in $\mathbb{R}^n$ (or in $\mathbb{C}^n$):

$$S_F^{n-1} = \{\mathbf{u} \in F^n \,|\, \|\mathbf{u}\| = 1\}$$

For example, the sphere in $\mathbb{R}^2$ is the unit circle (it is a curve, it has dimension 1), the sphere in $\mathbb{R}^3$ is the unit sphere (it is a surface, it has dimension 2); for higher dimensions we need to use our imagination.

### 5.2. Extrema of the Rayleigh's quotient.

5.2.1. *Closed sets, bounded sets, compact sets.* You probably know very well the extreme value theorem for continuous function on the real line:

**Theorem 50. The extreme value theorem in dimension one.**
*A functions $f(x)$ which is continuous on a closed and bounded interval $[a, b]$ has a maximum value (and a minimum value) on $[a, b]$.*

To formulate an analogue of this theorem in higher dimensions we need to specify what we mean by a *closed* set and by a *bounded* set.

**Definition 51.** *A set $S$ is called* **closed** *if it contains all its limit points: if a sequence of points in $S$, $\{\mathbf{x}_k\}_k \subset S$ converges, $\lim_{k\to\infty} = \mathbf{x}$, then the limit $\mathbf{x}$ is also in $S$.*

For example, the intervals $[2, 6]$ and $[2, +\infty)$ are closed in $\mathbb{R}$, but $[2, 6)$ is not closed. The closed unit disk $\{\mathbf{x} \in \mathbb{R}^2 \,|\, \|\mathbf{x}\| \leq 1\}$ is closed in $\mathbb{R}^2$, but the punctured disk $\{\mathbf{x} \in \mathbb{R}^2 \,|\, 0 < \|\mathbf{x}\| \leq 1\}$ or the open disk $\{\mathbf{x} \in \mathbb{R}^2 \,|\, \|\mathbf{x}\| < 1\}$ are not closed sets.

**Definition 52.** *A set $S$ is called* **bounded** *if there is a number larger than all the lengths of the vectors in $S$: there is $M > 0$ so that $\|\mathbf{x}\| \leq M$ for all $\mathbf{x} \in S$.*

For example, the intervals $[2, 6]$ and $[2, 6)$ are bounded in $\mathbb{R}$, but $[2, +\infty)$ is not. The square $\{\mathbf{x} \in \mathbb{R}^2 \,|\, |x_1| < 1, \text{ and } |x_2| < 1\}$ is bounded in $\mathbb{R}^2$, but the vertical strip $\{\mathbf{x} \in \mathbb{R}^2 \,|\, |x_1| < 1\}$ is not.

**Theorem 53. The extreme value theorem in finite dimensions.**
*A functions $f(x)$ which is continuous on a closed and bounded set $S$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a maximum value (and a minimum value) on $S$.*

In infinite dimensions Theorem 53 is not true in this form. A more stringent condition on the set $S$ is needed to ensure existence of extreme values of continuous functions on $S$ (the set must be *compact*).

It is intuitively clear (and rigorously proved in mathematical analysis) that any sphere in $F^n$ is a closed and bounded set.

5.2.2. *Minimum and maximum of the Rayleigh's quotient.* The Rayleigh's quotient calculated on unit vectors is a quadratic polynomial, and therefore it is a continuous function on the unit sphere, and therefore

**Proposition 54.** *The Rayleigh's quotient has an absolute maximum and a minimum.*

What happens if $A$ is not self-adjoint? Recall that the quadratic form $\langle \mathbf{x}, A\mathbf{x}\rangle$ has the same value if we replace $A$ by its self-adjoint part, $\frac{1}{2}(A+A^*)$, therefore the Rayleigh's quotient of $A$ is the same as he Rayleigh's quotient of the self-adjoint part of $A$ (hence information about $A$ is lost).

The extreme values of the Rayleigh's quotient can be easily linked to the eigenvalues of the self-adjoint matrix $A$. To see this, diagonalize the quadratic form $\langle \mathbf{x}, A\mathbf{x}\rangle$: consider a unitary matrix $U$ which diagonalizes $A$:

$$U^* A U = \Lambda = diag(\lambda_1, \ldots, \lambda_n)$$

In the new coordinates $\mathbf{y} = U^*\mathbf{x}$ we have

$$\langle \mathbf{x}, A\mathbf{x}\rangle = \langle \mathbf{x}, U\Lambda U^*\mathbf{x}\rangle = \langle U^*\mathbf{x}, \Lambda U^*\mathbf{x}\rangle = \langle \mathbf{y}, \Lambda\mathbf{y}\rangle = \sum_{j=1}^{n} \lambda_j |y_j|^2$$

which together with $\|\mathbf{x}\| = \|U\mathbf{y}\| = \|\mathbf{y}\|$ gives for the Rayleigh's quotient

$$R(\mathbf{x}) = R(U\mathbf{y}) = \frac{\sum_{j=1}^{n} \lambda_j |y_j|^2}{\|\mathbf{y}\|^2} = \sum_{j=1}^{n} \lambda_j \frac{|y_j|^2}{\|\mathbf{y}\|^2} \equiv R_U(\mathbf{y}) \tag{37}$$

Since $A$ is self-adjoint, its eigenvalues $\lambda_j$ are real; assume them ordered in an increasing sequence:

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

Then clearly

$$\sum_{j=1}^{n} \lambda_j |y_j|^2 \leq \lambda_n \sum_{j=1}^{n} |y_j|^2 = \lambda_n \|\mathbf{y}\|^2$$

and

$$\sum_{j=1}^{n} \lambda_j |y_j|^2 \geq \lambda_1 \sum_{j=1}^{n} |y_j|^2 = \lambda_1 \|\mathbf{y}\|^2$$

therefore

$$\lambda_1 \leq R(\mathbf{x}) \leq \lambda_n \quad \text{for all } \mathbf{x} \neq \mathbf{0}$$

Equalities are attained since $R_U(\mathbf{e}_1) = \lambda_1$ and $R_U(\mathbf{e}_n) = \lambda_n$. Going to coordinates $\mathbf{x}$ minimum is attained for $\mathbf{x} = U\mathbf{e}_1 = \mathbf{v}_1 =$ eigenvector corresponding to $\lambda_1$ since $R(\mathbf{v}_1) = R_U(\mathbf{e}_1) = \lambda_1$ , and for $\mathbf{x} = U\mathbf{v}_n = \mathbf{v}_n =$ eigenvector corresponding to $\lambda_n$, maximum is attained since $R(\mathbf{v}_n) = R_U(\mathbf{e}_n) = \lambda_n$. This proves:

**Theorem 55.** *If $A$ is a self-adjoint matrix then*

$$\max \frac{\langle \mathbf{x}, A\mathbf{x}\rangle}{\|\mathbf{x}\|^2} = \lambda_n \textit{ the max eigenvalue of } A, \textit{ attained for } \mathbf{x} = \mathbf{v}_n$$

*and*

$$\min \frac{\langle \mathbf{x}, A\mathbf{x}\rangle}{\|\mathbf{x}\|^2} = \lambda_1 \textit{ the min eigenvalue of } A, \textit{ attained for } \mathbf{x} = \mathbf{v}_1$$

As an important consequence in numerical calculations: the maximum eigenvalue of $A$ can be found by solving a maximization problem, and the minimum eigenvalue - by a minimization problem.

5.3. **The minimax principle.** Reducing the dimension of $A$ we can find all the eigenvalues, one by one. This reduction of the dimension is done using:

5.3.1. *Spitting of the space under the action of a self-adjoint matrix.* Any matrix leaves invariant its eigenspaces. Since eigenvectors of a self-adjoint matrix $A$ form an orthogonal set, then self-adjoint matrices leaves invariant their orthogonal complement as well:

**Remark.** *Let $A$ be any $n \times n$ self-adjoint matrix.*

*a) The vector space $F^n$ splits*

$$F^n = \oplus_{\lambda \in \sigma(A)} V_\lambda$$

*where each eigenspace $V_\lambda$ is invariant under $A$, and so is its orthogonal complement, which equals the direct sum of all the other eigenspaces:*

$$V_\lambda^\perp = \oplus_{\lambda' \in \sigma(A), \lambda' \neq \lambda} \quad V_{\lambda'}$$

*b) If $\mathbf{v}$ is an eigenvector then $A$ leaves invariant $Sp(\mathbf{v})$ and $Sp(\mathbf{v})^\perp$.*

*Proof:*

Only part b) needs a proof.

$A(Sp(\mathbf{v})) \subset Sp(\mathbf{v})$ because if $\mathbf{x} \in Sp(\mathbf{v})$ then $\mathbf{x} = x\mathbf{v}$ therefore $A\mathbf{x} = cA\mathbf{v} = c\lambda\mathbf{v} \in Sp(\mathbf{v})$.

$A(Sp(\mathbf{v})^\perp) \subset Sp(\mathbf{v})^\perp$ because if $\mathbf{y} \in Sp(\mathbf{v})^\perp$ then this means that $\langle \mathbf{y}, \mathbf{v}\rangle = 0$. Then $\langle A\mathbf{y}, \mathbf{v}\rangle = \langle \mathbf{y}, A^*\mathbf{v}\rangle = \langle \mathbf{y}, A\mathbf{v}\rangle = \langle \mathbf{y}, \lambda\mathbf{v}\rangle = \lambda\langle \mathbf{y}, \mathbf{v}\rangle = 0$. □

5.3.2. *Minimax and maximin.* We saw that $\max R(\mathbf{x}) = \lambda_n = R(\mathbf{v}_n)$. Then the matrix $A$, as a linear transformation of the $n-1$ dimensional vector space $Sp(\mathbf{v}_n)^\perp$ to itself has its largest eigenvalue $\lambda_{n-1}$ (we reduced the dimension!). Then

$$\max_{\mathbf{x}\in Sp(\mathbf{v}_n)^\perp} R(\mathbf{x}) = \lambda_{n-1} \quad \text{is attained for } \mathbf{x} = \mathbf{v}_{n-1}$$

Note that the statement $\mathbf{x} \in Sp(\mathbf{v}_n)^\perp$ can be formulated as the constraint $\langle \mathbf{x}, \mathbf{v}_n\rangle = 0$:

$$\max_{\langle \mathbf{x},\mathbf{v}_n\rangle=0} R(\mathbf{x}) = \lambda_{n-1}$$

We can do even better: we can obtain $\lambda_{n-1}$ without knowing $\mathbf{v}_n$ or $\lambda_n$. To achieve this, subject $\mathbf{x}$ to *any* constraint: $\langle \mathbf{x}, \mathbf{z}\rangle = 0$ for some $\mathbf{z} \neq \mathbf{0}$.

It is easier to see what happens in coordinates $\mathbf{y} = U^*\mathbf{x}$ in which $A$ is diagonal. The constraint $\langle \mathbf{x}, \mathbf{z}\rangle = 0$ is equivalent to $\langle \mathbf{y}, \mathbf{w}\rangle = 0$ where $\mathbf{w} = U\mathbf{z}$ is some nonzero vector. On one hand, obviously

$$\max_{\mathbf{y}:\,\langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) \leq \lambda_n$$

$$\max_{\mathbf{y}:\,\langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) \geq \lambda_{n-1} \tag{38}$$

which implies that

$$\min_{\mathbf{w}\neq\mathbf{0}} \max_{\mathbf{y}:\,\langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) \geq \lambda_{n-1} \tag{39}$$

We now argue that equality is atained in (39) for special $\mathbf{w}$ and $\mathbf{y}$. Indeed, consider a nonzero vector $\tilde{\mathbf{y}} = (0, \ldots, 0, y_{n-1}, y_n)^T$ with $\langle \tilde{\mathbf{y}}, \mathbf{w}\rangle = 0$. ($\tilde{\mathbf{y}}$ is easy to find: if $w_n \neq 0$ take $y_{n-1} = 1$ and $y_n = -w_{n-1}/w_n$, and if $w_n = 0$ take $y_{n-1} = 0,\ y_n = 1$).

Using formula (37)

$$R_U(\tilde{\mathbf{y}}) = \frac{\lambda_{n-1}|y_{n-1}|^2 + \lambda_n|y_n|^2}{|y_{n-1}|^2 + |y_n|^2} \geq \frac{\lambda_{n-1}|y_{n-1}|^2 + \lambda_{n-1}|y_n|^2}{|y_{n-1}|^2 + |y_n|^2} = \lambda_{n-1}$$

Since for $\mathbf{w} = \mathbf{e}_n$ we have equality:

$$\max_{\mathbf{y}:\,\langle \mathbf{y},\mathbf{e}_n\rangle=0} R_U(\mathbf{y}) = \lambda_{n-1}$$

then in (39) there is equality

$$\min_{\mathbf{w}\neq\mathbf{0}} \max_{\mathbf{y}:\,\langle \mathbf{y},\mathbf{w}\rangle=0} R_U(\mathbf{y}) = \lambda_{n-1}$$

In a similar way it is shown that $\lambda_{n-2}$ is obtained by a minimum-maximum process, but with two constraints:

$$\min_{\mathbf{w}_1,\mathbf{w}_2\neq\mathbf{0}} \max_{\substack{\langle \mathbf{y},\mathbf{w}_1\rangle=0\\ \langle \mathbf{y},\mathbf{w}_2\rangle=0}} R_U(\mathbf{y}) = \lambda_{n-2} \tag{40}$$

Indeed, consider a nonzero vector $\tilde{\mathbf{y}} = (0,\ldots,0,y_{n-2},y_{n-1},y_n)^T$ satisfying $\langle \tilde{\mathbf{y}},\mathbf{w}_1\rangle = 0$ and $\langle \tilde{\mathbf{y}},\mathbf{w}_2\rangle = 0$. Then in formula (37)

$$R_U(\tilde{\mathbf{y}}) = \frac{\lambda_{n-2}|y_{n-2}|^2 + \lambda_{n-1}|y_{n-1}|^2 + \lambda_n|y_n|^2}{|y_{n-2}|^2 + |y_{n-1}|^2 + |y_n|^2}$$
$$\geq \frac{\lambda_{n-2}|y_{n-2}|^2 + \lambda_{n-2}|y_{n-1}|^2 + \lambda_{n-2}|y_n|^2}{|y_{n-2}|^2 + |y_{n-1}|^2 + |y_n|^2} = \lambda_{n-2}$$

which shows that

$$\max_{\substack{\langle \mathbf{y},\mathbf{w}_1\rangle=0\\ \langle \mathbf{y},\mathbf{w}_2\rangle=0}} R_U(\mathbf{y}) \geq \lambda_{n-2} \tag{41}$$

Since for $\mathbf{w}_1 = \mathbf{e}_n$ and $\mathbf{w}_2 = \mathbf{e}_{n-1}$ we have equality in (41), this implies (40).

Step by step, adding one extra constraint, the minimax procedure yields the next largest eigenvalue.

Going back to the variable $\mathbf{x}$ it is found that:

**Theorem 56. The minimax principle**

*Let $A$ be a self-adjoint matrix, with its eigenvalues numbered in an increasing sequence:*

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

*corresponding to the eigenvectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$.*

*Then its Rayleigh's quotient*

$$R(\mathbf{x}) = \frac{\langle \mathbf{x}, A\mathbf{x}\rangle}{\|\mathbf{x}\|^2}$$

*satisfies*

$$\max_{\mathbf{x}\neq 0} R(\mathbf{x}) = \lambda_n$$

$$\min_{\mathbf{z}\neq\mathbf{0}} \max_{\langle \mathbf{x},\mathbf{z}\rangle=0} R(\mathbf{x}) = \lambda_{n-1}$$

$$\min_{\mathbf{z}_1,\mathbf{z}_2\neq\mathbf{0}} \max_{\substack{\langle \mathbf{x},\mathbf{z}_1\rangle=0\\ \langle \mathbf{x},\mathbf{z}_2\rangle=0}} R(\mathbf{x}) = \lambda_{n-2}$$

*and in general*

$$\min_{\mathbf{z}_1,\ldots\mathbf{z}_k\neq\mathbf{0}} \max_{\substack{\langle \mathbf{x},\mathbf{z}_1\rangle=0\\ \vdots\\ \langle \mathbf{x},\mathbf{z}_k\rangle=0}} R(\mathbf{x}) = \lambda_{n-k}, \qquad k=1,2,\ldots,n-1$$

**Remark.** Sometimes the minimax principle is formulated as

$$\min_{V_j} \max_{\mathbf{x}\in V_j} R(\mathbf{x}) = \lambda_j, \quad j = 1, \ldots, n$$

where $V_j$ denotes an arbitrary subspace of dimension $j$.

The two formulations are equivalent since the set

$$V_{n-k} = \{\mathbf{x} \,|\, \langle \mathbf{x}, \mathbf{z}_1\rangle = 0, \ldots, \langle \mathbf{x}, \mathbf{z}_k\rangle = 0\}$$

is a vector space of dimension $n-k$ if $\mathbf{z}_1, \ldots, \mathbf{z}_k$ are linearly independent.

A similar construction starting with the lowest eigenvalue produces:

**Theorem 57. The maximin principle**

*Under the assumptions of Theorem 56*

$$\min_{\mathbf{x}\neq 0} R(\mathbf{x}) = \lambda_1$$

$$\max_{\mathbf{z}\neq \mathbf{0}} \min_{\langle \mathbf{x},\mathbf{z}\rangle = 0} R(\mathbf{x}) = \lambda_2$$

*and in general*

$$\max_{\mathbf{z}_1,\ldots\mathbf{z}_k\neq \mathbf{0}} \min_{\substack{\langle \mathbf{x},\mathbf{z}_1\rangle = 0 \\ \vdots \\ \langle \mathbf{x},\mathbf{z}_k\rangle = 0}} R(\mathbf{x}) = \lambda_{k+1}, \qquad k = 1, 2, \ldots, n-1$$

5.4. **The minimax principle for the generalized eigenvalue problem.** Suppose $\lambda_1 \leq \lambda_1 \leq \ldots \leq \lambda_n$ are eigenvalues for the problem

$$A\mathbf{v} = \lambda B\mathbf{v}, \quad A \text{ symmetric}, \; B \text{ positive definite} \tag{42}$$

It was seen in §4.8 that if $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is the matrix whose columns are the generalized eigenvectors of the problem (42), then both matrices $A$ and $B$ are diagonalized using a congruence transformation: $S^TAS = \Lambda$ and $S^TBS = I$.

Defining

$$R(\mathbf{x}) = \frac{\langle \mathbf{x}, A\mathbf{x}\rangle}{\langle \mathbf{x}, B\mathbf{x}\rangle}$$

it is found that in coordinates $\mathbf{x} = S\mathbf{y}$:

$$R(\mathbf{x}) = R(S\mathbf{y}) = \frac{\langle S\mathbf{y}, AS\mathbf{y}\rangle}{\langle S\mathbf{y}, BS\mathbf{y}\rangle} = \frac{\langle \mathbf{y}, S^TAS\mathbf{y}\rangle}{\langle \mathbf{y}, S^TBS\mathbf{y}\rangle} = \frac{\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2}{y_1^2 + \ldots + y_n^2}$$

and therefore

$$\max R(\mathbf{x}) = \lambda_n, \quad \min R(\mathbf{x}) = \lambda_1$$