5. The Rayleigh’s principle and the minimax principle for the eigenvalues of a self-adjoint matrix

Eigenvalues of self-adjoint matrices are easy to calculate. This section shows how this is done using a minimization, or maximization procedure.

5.1. The Rayleigh’s quotient.

Definition 49. Let $A = A^*$ be a self-adjoint matrix. The Rayleigh’s quotient is the function

$$R(x) = \frac{\langle x, Ax \rangle}{\|x\|^2}, \quad \text{for } x \neq 0$$

Note that

$$R(x) = \frac{x}{\|x\|}, A \frac{x}{\|x\|} = \langle u, Au \rangle \quad \text{where } u = \frac{x}{\|x\|}$$

so in fact, it suffices to define the Rayleigh’s quotient on unit vectors.

The set of unit vectors in $\mathbb{R}^n$ (or in $\mathbb{C}^n$), is called the $n-1$ dimensional sphere in $\mathbb{R}^n$ (or in $\mathbb{C}^n$):

$$S^{n-1}_F = \{ u \in F^n | \|u\| = 1 \}$$

For example, the sphere in $\mathbb{R}^2$ is the unit circle (it is a curve, it has dimension 1), the sphere in $\mathbb{R}^3$ is the unit sphere (it is a surface, it has dimension 2); for higher dimensions we need to use our imagination.

5.2. Extrema of the Rayleigh’s quotient.

5.2.1. Closed sets, bounded sets, compact sets. You probably know very well the extreme value theorem for continuous function on the real line:

Theorem 50. The extreme value theorem in dimension one.

A function $f(x)$ which is continuous on a closed and bounded interval $[a, b]$ has a maximum value (and a minimum value) on $[a, b]$.

To formulate an analogue of this theorem in higher dimensions we need to specify what we mean by a closed set and by a bounded set.

Definition 51. A set $S$ is called closed if it contains all its limit points: if a sequence of points in $S$, $\{x_k\}_k \subset S$ converges, $\lim_{k \to \infty} x_k = x$, then the limit $x$ is also in $S$.

For example, the intervals $[2, 6]$ and $[2, +\infty)$ are closed in $\mathbb{R}$, but $[2, 6)$ is not closed. The closed unit disk $\{ x \in \mathbb{R}^2 | \|x\| \leq 1 \}$ is closed in $\mathbb{R}^2$, but the punctured disk $\{ x \in \mathbb{R}^2 | 0 < \|x\| \leq 1 \}$ or the open disk $\{ x \in \mathbb{R}^2 | \|x\| < 1 \}$ are not closed sets.

Definition 52. A set $S$ is called bounded if there is a number larger than all the lengths of the vectors in $S$: there is $M > 0$ so that $\|x\| \leq M$ for all $x \in S$. 
For example, the intervals \([2, 6]\) and \([2, 6)\) are bounded in \(\mathbb{R}\), but \([2, +\infty)\) is not. The square \(\{x \in \mathbb{R}^2 \mid |x_1| < 1, \text{ and } |x_2| < 1\}\) is bounded in \(\mathbb{R}^2\), but the vertical strip \(\{x \in \mathbb{R}^2 \mid |x_1| < 1\}\) is not.

**Theorem 53.** The extreme value theorem in finite dimensions.

A function \(f(x)\) which is continuous on a closed and bounded set \(S\) in \(\mathbb{R}^n\) or \(\mathbb{C}^n\) has a maximum value (and a minimum value) on \(S\).

In infinite dimensions Theorem 53 is not true in this form. A more stringent condition on the set \(S\) is needed to ensure existence of extreme values of continuous functions on \(S\) (the set must be compact).

It is intuitively clear (and rigorously proved in mathematical analysis) that any sphere in \(\mathbb{F}^n\) is a closed and bounded set.

**5.2.2. Minimum and maximum of the Rayleigh’s quotient.** The Rayleigh’s quotient calculated on unit vectors is a quadratic polynomial, and therefore it is a continuous function on the unit sphere, and therefore

**Proposition 54.** The Rayleigh’s quotient has an absolute maximum and a minimum.

What happens if \(A\) is not self-adjoint? Recall that the quadratic form \(\langle x, Ax \rangle\) has the same value if we replace \(A\) by its self-adjoint part, \(\frac{1}{2}(A+A^*)\), therefore the Rayleigh’s quotient of \(A\) is the same as the Rayleigh’s quotient of the self-adjoint part of \(A\) (hence information about \(A\) is lost).

The extreme values of the Rayleigh’s quotient can be easily linked to the eigenvalues of the self-adjoint matrix \(A\). To see this, diagonalize the quadratic form \(\langle x, Ax \rangle\): consider a unitary matrix \(U\) which diagonalizes \(A\):

\[U^*AU = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)\]

In the new coordinates \(y = U^*x\) we have

\[\langle x, Ax \rangle = \langle x, U\Lambda U^*x \rangle = \langle U^*x, \Lambda U^*x \rangle = \langle y, \Lambda y \rangle = \sum_{j=1}^{n} \lambda_j |y_j|^2\]

which together with \(\|x\| = \|Uy\| = \|y\|\) gives for the Rayleigh’s quotient

\[R(x) = R(Uy) = \sum_{j=1}^{n} \lambda_j |y_j|^2 / \|y\|^2 = \sum_{j=1}^{n} \lambda_j |y_j|^2 / \|y\|^2 = R_U(y)\]

Since \(A\) is self-adjoint, its eigenvalues \(\lambda_j\) are real; assume them ordered in an increasing sequence:

\[\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n\]
Then clearly
\[ \sum_{j=1}^{n} \lambda_j |y_j|^2 \leq \lambda_n \sum_{j=1}^{n} |y_j|^2 = \lambda_n \|y\|^2 \]
and
\[ \sum_{j=1}^{n} \lambda_j |y_j|^2 \geq \lambda_1 \sum_{j=1}^{n} |y_j|^2 = \lambda_1 \|y\|^2 \]
therefore
\[ \lambda_1 \leq R(x) \leq \lambda_n \quad \text{for all } x \neq 0 \]
Equalities are attained since \( R_U(e_1) = \lambda_1 \) and \( R_U(e_n) = \lambda_n \). Going to coordinates \( x \) minimum is attained for \( x = Ue_1 = v_1 \) = eigenvector corresponding to \( \lambda_1 \) since \( R(v_1) = R_U(e_1) = \lambda_1 \), and for \( x = Uv_n = v_n \) = eigenvector corresponding to \( \lambda_n \), maximum is attained since \( R(v_n) = R_U(e_n) = \lambda_n \). This proves:

**Theorem 55.** If \( A \) is a self-adjoint matrix then
\[ \max \frac{\langle x, Ax \rangle}{\|x\|^2} = \lambda_n \quad \text{the max eigenvalue of } A, \text{ attained for } x = v_n \]
and
\[ \min \frac{\langle x, Ax \rangle}{\|x\|^2} = \lambda_1 \quad \text{the min eigenvalue of } A, \text{ attained for } x = v_1 \]

As an important consequence in numerical calculations: the maximum eigenvalue of \( A \) can be found by solving a maximization problem, and the minimum eigenvalue - by a minimization problem.

### 5.3. The minimax principle.
Reducing the dimension of \( A \) we can find all the eigenvalues, one by one. This reduction of the dimension is done using:

#### 5.3.1. Spitting of the space under the action of a self-adjoint matrix.
Any matrix leaves invariant its eigenspaces. Since eigenvectors of a self-adjoint matrix \( A \) form an orthogonal set, then self-adjoint matrices leaves invariant their orthogonal complement as well:

**Remark.** Let \( A \) be any \( n \times n \) self-adjoint matrix.

a) The vector space \( \mathbb{F}^n \) splits
\[ \mathbb{F}^n = \bigoplus_{\lambda \in \sigma(A)} V_\lambda \]
where each eigenspace \( V_\lambda \) is invariant under \( A \), and so is its orthogonal complement, which equals the direct sum of all the other eigenspaces:
\[ V_\lambda^\perp = \bigoplus_{\lambda' \in \sigma(A), \lambda' \neq \lambda} V_{\lambda'} \]
b) If \( v \) is an eigenvector then \( A \) leaves invariant \( Sp(v) \) and \( Sp(v)^\perp \).

**Proof:**
Only part b) needs a proof.
A(Sp(v)) ⊆ Sp(v) because if x ∈ Sp(v) then x = xv therefore Ax = cAv = cλv ∈ Sp(v).

A(Sp(v)⊥) ⊆ Sp(v)⊥ because if y ∈ Sp(v)⊥ then this means that ⟨y, v⟩ = 0. Then ⟨Ay, v⟩ = ⟨y, Av⟩ = ⟨y, λv⟩ = λ⟨y, v⟩ = 0. □

5.3.2. Minimax and maximin. We saw that max R(x) = λn = R(vn). Then the matrix A, as a linear transformation of the n − 1 dimensional vector space Sp(vn)⊥ to itself has its largest eigenvalue λn−1 (we reduced the dimension!). Then

\[
\max_{x ∈ Sp(v_n)⊥} R(x) = \lambda_{n-1} \quad \text{is attained for } x = v_{n-1}
\]

Note that the statement x ∈ Sp(vn)⊥ can be formulated as the constraint

\[
\max_{\langle x, v_n \rangle = 0} R(x) = \lambda_{n-1}
\]

We can do even better: we can obtain λn−1 without knowing v_n or λ_n. To achieve this, subject x to any constraint: ⟨x, z⟩ = 0 for some z ≠ 0.

It is easier to see what happens in coordinates y = Ux in which A is diagonal. The constraint ⟨x, z⟩ = 0 is equivalent to ⟨y, w⟩ = 0 where w = Uz is some nonzero vector. On one hand, obviously

\[
\max_{y: \langle y, w \rangle = 0} R_U(y) \leq \lambda_n
\]

(38)

\[
\max_{y: \langle y, w \rangle = 0} R_U(y) \geq \lambda_{n-1}
\]

which implies that

(39)

\[
\min_{w \neq 0} \max_{y: \langle y, w \rangle = 0} R_U(y) \geq \lambda_{n-1}
\]

We now argue that equality is attained in (39) for special w and y. Indeed, consider a nonzero vector \( \tilde{y} = (0, \ldots, 0, y_{n-1}, y_n)^T \) with \( \langle \tilde{y}, w \rangle = 0 \). \( \tilde{y} \) is easy to find: if \( w_n \neq 0 \) take \( y_{n-1} = 1 \) and \( y_n = -w_{n-1}/w_n \), and if \( w_n = 0 \) take \( y_{n-1} = 0 \), \( y_n = 1 \).

Using formula (37)

\[
R_U(\tilde{y}) = \frac{\lambda_{n-1}|y_{n-1}|^2 + \lambda_n|y_n|^2}{|y_{n-1}|^2 + |y_n|^2} \geq \frac{\lambda_{n-1}|y_{n-1}|^2 + \lambda_{n-1}|y_n|^2}{|y_{n-1}|^2 + |y_n|^2} = \lambda_{n-1}
\]

Since for w = e_n we have equality:

\[
\max_{y: \langle y, e_n \rangle = 0} R_U(y) = \lambda_{n-1}
\]

then in (39) there is equality

\[
\min_{w \neq 0} \max_{y: \langle y, w \rangle = 0} R_U(y) = \lambda_{n-1}
\]
In a similar way it is shown that $\lambda_{n-2}$ is obtained by a minimum-maximum process, but with two constraints:

\[
\min_{w_1, w_2 \neq 0} \max_{(y, w_1)} R_U(y) = \lambda_{n-2}
\]

Indeed, consider a nonzero vector $\tilde{y} = (0, \ldots, 0, y_{n-2}, y_{n-1}, y_n)^T$ satisfying $\langle \tilde{y}, w_1 \rangle = 0$ and $\langle \tilde{y}, w_2 \rangle = 0$. Then in formula (37)

\[
R_U(\tilde{y}) = \frac{\lambda_{n-2}|y_{n-2}|^2 + \lambda_{n-1}|y_{n-1}|^2 + \lambda_n|y_n|^2}{|y_{n-2}|^2 + |y_{n-1}|^2 + |y_n|^2}
\]

\[
\geq \frac{\lambda_{n-2}|y_{n-2}|^2 + \lambda_{n-1}|y_{n-1}|^2 + \lambda_n|y_n|^2}{|y_{n-2}|^2 + |y_{n-1}|^2 + |y_n|^2} = \lambda_{n-2}
\]

which shows that

\[
\max_{(y, w_1)} R_U(y) \geq \lambda_{n-2}
\]

Since for $w_1 = e_n$ and $w_2 = e_{n-1}$ we have equality in (41), this implies (40).

Step by step, adding one extra constraint, the minimax procedure yields the next largest eigenvalue.

Going back to the variable $x$ it is found that:

**Theorem 56. The minimax principle**

Let $A$ be a self-adjoint matrix, with its eigenvalues numbered in an increasing sequence:

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

corresponding to the eigenvectors $v_1, \ldots, v_n$.

Then its Rayleigh’s quotient

$$R(x) = \frac{\langle x, Ax \rangle}{\|x\|^2}$$

satisfies

\[
\max_{x \neq 0} R(x) = \lambda_n
\]

\[
\min_{x \neq 0} \max_{(x, z) = 0} R(x) = \lambda_{n-1}
\]

\[
\min_{z_1, z_2 \neq 0} \max_{\langle x, z_1 \rangle = 0} \max_{\langle x, z_2 \rangle = 0} R(x) = \lambda_{n-2}
\]

and in general

\[
\min_{z_1, \ldots, z_k \neq 0} \max_{\langle x, z_1 \rangle = 0, \ldots, \langle x, z_k \rangle = 0} R(x) = \lambda_{n-k}, \quad k = 1, 2, \ldots, n - 1
\]
Remark. Sometimes the minimax principle is formulated as
\[
\min_{V_j} \max_{x \in V_j} R(x) = \lambda_j, \quad j = 1, \ldots, n
\]
where \(V_j\) denotes an arbitrary subspace of dimension \(j\).

The two formulations are equivalent since the set
\[
V_{n-k} = \{ x \mid \langle x, z_1 \rangle = 0, \ldots, \langle x, z_k \rangle = 0 \}
\]
is a vector space of dimension \(n - k\) if \(z_1, \ldots, z_k\) are linearly independent.

A similar construction starting with the lowest eigenvalue produces:

**Theorem 57. The maximin principle**

Under the assumptions of Theorem 56

\[
\min_{x \neq 0} R(x) = \lambda_1
\]

\[
\max_{z \neq 0} \min_{\langle x, z \rangle = 0} R(x) = \lambda_2
\]

and in general\[
\max_{z_1, \ldots, z_k \neq 0} \min_{\langle x, z_1 \rangle = 0} \ldots \min_{\langle x, z_k \rangle = 0} R(x) = \lambda_{k+1}, \quad k = 1, 2, \ldots, n - 1
\]

5.4. **The minimax principle for the generalized eigenvalue problem.**

Suppose \(\lambda_1 \leq \lambda_1 \leq \ldots \leq \lambda_n\) are eigenvalues for the problem

(42) \( Av = \lambda Bv, \; A \text{ symmetric, } B \text{ positive definite} \)

It was seen in §4.8 that if \(S = [v_1, \ldots, v_n]\) is the matrix whose columns are the generalized eigenvectors of the problem (42), then both matrices \(A\) and \(B\) are diagonalized using a congruence transformation: \(S^T AS = \Lambda\) and \(S^T BS = I\).

Defining

\[
R(x) = \frac{\langle x, Ax \rangle}{\langle x, Bx \rangle}
\]
it is found that in coordinates \(x = Sy\):

\[
R(x) = R(Sy) = \frac{\langle Sy, ASy \rangle}{\langle Sy, BSy \rangle} = \frac{\langle y, S^T ASy \rangle}{\langle y, S^T BSy \rangle} = \frac{\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2}{y_1^2 + \ldots + y_n^2}
\]

and therefore

\[
\max R(x) = \lambda_n, \quad \min R(x) = \lambda_1
\]