Theorem 2. *The operator*

$$L = \frac{1}{w(x)}\left(-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right) \tag{25}$$

is self-adjoint in the weighted (real) Hilbert space $H = L^2([a,b], w(x)dx)$ *on the domain*

$$D = \{u \in H \,|\, u', u'' \in H,\ B_a[u] = 0,\ B_b[u] = 0\} \tag{26}$$

where $B_a[u]$, $B_b[u]$ *are any of the boundary conditions listed above.*

Proof. To show that $\langle Lu, v\rangle - \langle u, Lv\rangle = 0$ for all $u, v \in D$ we use, as usually integration by parts. This work was done by the Green's identity, which gives (noting that the terms containing q cancel each other):

$$\langle Lu, v\rangle - \langle u, Lv\rangle = \int_a^b \frac{1}{w}\left[-\left(pu'\right)' + qu\right] v\, w\, dx - \int_a^b u\,\frac{1}{w}\left[-\left(pv'\right)' + qv\right] w\, dx$$

$$= -p(u'v - uv')\Big|_a^b = p(a)\,(u'v - uv')\Big|_{x=a} - p(b)\,(u'v - uv')\Big|_{x=b} \tag{27}$$

We have that

$$p(a)(u'v - uv')\Big|_{x=a} = 0,\ p(b)(u'v - uv')\Big|_{x=b} = 0 \tag{28}$$

because $B_a[u] = 0 = B_a[v]$ and $B_b[u] = 0 = B_b[v]$.

Indeed, if $\alpha' = 0$ then $u(a) = 0 = v(a)$ therefore $(u'v - uv')\Big|_{x=a} = 0$, and if $\alpha' \neq 0$ then $u'(a) = -\alpha/\alpha' u(a)$, $v'(a) = -\alpha/\alpha' v(a)$ which substituted into the first relation of (28) gives again zero. The second relation of (28) follows in a similar way. □

2.6. **Conclusions.** Since L is selfadjoint on D, this implies that its eigenvalues (if any!) are real, and that eigenfunctions corresponding to different eigenvalues are orthogonal.

We will show that indeed, there exist infinitely many eigenvalues λ_n, and that the eigenfunctions u_n form a complete set in the Hilbert space $L^2([a,b], w(x)dx)$.

We will accomplish this program by studying the solutions of the differential equation.

It turns out that, in addition, λ_n can be ordered increasingly, and $\lambda_n \to \infty$, and that eigenfunctions u_n oscillate, and the larger n, the more rapid the oscillations.

3. Eigenfunctions associated to one eigenvalue

3.1. Regular problems.

Lemma 3. *For regular problems (19), (20) the eigenspaces are one-dimensional: there is a unique (up to a scalar multiple) eigenfunction associated to each eigenvalue.*

Proof. We show that the eigenspace associated to one eigenvalue of (19) is one dimensional: any two (nonzero) solutions $u_1(x)$, $u_2(x)$ of (19) (for the same λ) are linearly dependent.

Assume, to get a contradiction, that u_1 and u_2 are linearly independent. Then the general solution of the differential equation in (19) is $u = C_1u_1 + C_2u_2$ with C_1, C_2 arbitrary constants. Since the boundary conditions are linear, it follows that $B_a[u] = C_1B_a[u_1] + C_2B_a[u_2] = 0$, $B_b[u] = C_1B_b[u_1] + C_2B_b[u_2] = 0$ therefore the boundary conditions are satisfied by any solution of the differential equation. For u the solution with $u(a) = 1, u(a) = 0$ we find that $\alpha = 0$ and u the solution with $u(a) = 0, u'(a) = 1$ it follows that $\alpha' = 0$, which contradicts the nontriviality assumption on B_a. (A similiar argument can be made at $x = b$.)

Therefore, u_1 and u_2 must be linearly dependent, hence scalar multiples of each other. □

3.2. When $p(x)$ vanishes at one endpoint.

Suppose that $p(b) = 0$. Consider the the Sturm-Liouville problem (19), (22).

The Sturm-Liouville operator (25) is selfadjoint on the domain

$$D = \{u \in H \mid u', u'' \in H,\ B_a[u] = 0\} \tag{29}$$

Indeed, relations (28) hold: at $x = b$ because $p(b) = 0$ and at $x = a$ with the same proof as for Theorem 25.

The eigenspaces are still one-dimensionsl, as the proof of Lemma 3 works.

The singular case with $p(a) = 0$ is similar.

3.3. Periodic problems.

If $p(a) = p(b)$ the Sturm-Liouville eigenvalue problem (19), (23) is also selfadjoint: the operator (25) is selfadjoint on

$$D = \{u \in H \mid u', u'' \in H,\ C[u] = 0,\ C'[u] = 0\} \tag{30}$$

since the last quantity in (27) is clearly zero for u, v in the domain (30).

However, for periodic boundary conditions the eigenspaces may be one, or two-dimensional.

Indeed, repeating the argument of §3.1 we find that if for an eigenvalue λ, if there are two independent eigenfunctions then all the solutions of the ODE are periodic (but the space of solutions of a second order ODE is a two-dimensional vector space). This means that the eigenspace is either one-dimensional or two-dimensional, and in the latter case we can choose two orthogonal eigenfunctions.

Here are sufficient conditions which ensure completeness of eigenfunctions:

Theorem 4. Completeness of the eigenfunctions

Suppose that L is a self-adjoint operator in a Hilbert space H, defined on a domain D dense in H.

Assume that L has the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ with $\lambda_n \to \infty$, and that each eigenspace is finite dimensional, spanned by a set of orthgonal eigenvectors u_n.

Then the set of eigenvectors is complete: they form a basis for the Hilbert space H.

The proof of the theorem is postponed until §8. We will put it to good use first.

4. Fourier series

4.1. **Basic facts.** It is easy to solve the following periodic Sturm-Liouville problem:

$$u'' + \lambda u = 0, \qquad u(-\pi) = u(\pi),\ u'(-\pi) = u'(\pi) \tag{31}$$

It has the eigenvalues $\lambda_n = n^2,\ n = 0, 1, 2, \ldots$. For $n > 0$ there are two orthogonal eigenfunctions corresponding to $\lambda_n = n^2$: e^{inx} and e^{-inx}, For $n = 0$ the eigenfunction corresponding to $\lambda_0 = 0$ is the constant function, say 1.