Theorem 4. Completeness of the eigenfunctions

Suppose that L is a self-adjoint operator in a Hilbert space H, defined on a domain D dense in H.

Assume that L has the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ with $\lambda_n \to \infty$, and that each eigenspace is finite dimensional, spanned by a set of orthgonal eigenvectors u_n.

Then the set of eigenvectors is complete: they form a basis for the Hilbert space H.

The proof of the theorem is postponed until §5. We will put it to good use first.

4. Fourier series

4.1. **Basic facts.** It is easy to solve the following periodic Sturm-Liouville problem:

$$u'' + \lambda u = 0, \qquad u(-\pi) = u(\pi),\ u'(-\pi) = u'(\pi) \tag{31}$$

It has the eigenvalues $\lambda_n = n^2,\ n = 0, 1, 2, \ldots$. For $n > 0$ there are two orthogonal eigenfunctions corresponding to $\lambda_n = n^2$: e^{inx} and e^{-inx}, For $n = 0$ the eigenfunction corresponding to $\lambda_0 = 0$ is the constant function, say 1.

Applying Theorem 4 to $L = -\frac{d^2}{dx^2}$ in $H = L^2[-\pi, \pi]$, which is self-adjoint on

$$D = \{u \in H \,|\, u', u'' \in H,\ u(\pi) - u(-\pi) = 0,\ u'(\pi) - u'(-\pi) = 0\}$$

having the eigenvalues $0, 1, 1, 2^2, 2^2, \ldots$ it follows that its eigenfunctions e^{inx}, $n \in \mathbb{Z}$ form an orthogonal basis for $L^2[-\pi, \pi]$.

Sometimes it is preferable to work with real-valued functions. In this case, in each eigenspace $Sp(e^{inx}, e^{-inx})$ corresponding to $\lambda_n = n^2 > 0$ we choose a basis consisting of real function: $\sin(nx)$ and $\cos(nx)$ (which are orthogonal). Therefore also its eigenfunctions 1, $\sin(nx)$, $\cos(nx)$ for $n = 1, 2, \ldots$ form an orthogonal basis for $L^2[-\pi, \pi]$. Therefore:

Theorem 5. *Any function $f \in L^2[-\pi, \pi]$ can be expanded in a Fourier series:*

$$f = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx} \tag{32}$$

where

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$$

Also

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right] \tag{33}$$

where the Fourier coefficients a_n and b_n are given by the formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos(nx)\,dx\ ,\ \ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\,dx,$$

The series (32) *and* (33) *converge in the L^2-norm of $L^2[-\pi,\pi]$.*

Exercise. Write $\hat{f}_n$ in terms of a_n, b_n. Write a_n, b_n in terms of $\hat{f}_n$.

4.2. **Fourier series, sine series and cosine series.** Recall that the eigenfunctions of the Sturm-Liouville problem

$$u'' + \lambda u = 0,\ u(0) = 0,\ u(\pi) = 0 \quad \text{on } [0,\pi] \tag{34}$$

are $\sin(nx)$, for $n = 1, 2, \ldots$ and they correspond to the eigenvalues $\lambda_n = n^2$. Theorem 4 does apply and we obtain that any function in $L^2[0,\pi]$ can be expanded in a sine-series.

On the other hand, the eigenfunctions of the Sturm-Liouville problem with Newman conditions

$$v'' + \lambda v = 0,\ v'(0) = 0,\ v'(\pi) = 0 \quad \text{on } [0,\pi] \tag{35}$$

are $\cos(nx)$, for $n = 0, 1, 2, \ldots$ and they correspond to the eigenvalues $\lambda_n = n^2$. (Indeed, denoting $v' = u$ we obtain the problem (34)). By Theorem 4 it follows that $\cos(nx)$, for $n = 0, 1, 2, \ldots$ also form an orthogonal basis for $L^2[0,\pi]$.

Therefore, we can expand any *function on* $[0,\pi]$ in a sine-series

$$f = \sum_{n=1}^{\infty} b_n \sin(nx), \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx) f(x)\,dx \tag{36}$$

or as a cosine series

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad \text{where } a_n = \frac{2}{\pi}\int_0^{\pi} \cos(nx) f(x)\,dx \tag{37}$$

where the convergence of (36) and (37) is in the L^2-norm on $[0,\pi]$ (i.e. in square average).

How do the series (33), (36), (37) reconcile each other?

The most important difference to keep in mind between (33) on one hand, and (36), (37) on the other hand, is that in the first case, of a bone-fide Fourier series, the function f is considered on an interval equal to the period of the functions sin and cos used, while in the case of sine, or cosine series, $f(x)$ is considered the interval of half-period length.

The connection between (33) and (36), (37) can be understood easily if we think in terms of even and odd functions. (This is one reason why it is helpful to choose an interval symmetric with respect to the origin.)

Recall that a function is called *even* if $f(-x) = f(x)$ for all x. For example the functions $1, x^2, x^4, x^8 - 3x^4, |x|, \cos(x)$ are even.

Recall that a function is called *odd* if $f(-x) = -f(x)$ for all x. For example the functions $x, x^3, x^5, x^9 - 2x, \sin(x), \tan x$ are odd.

4.2.1. *Functions on* $[-\pi, \pi]$.

Note: *If $f(x)$ is an even function then all $b_n = 0$ and therefore f has a cosine-series.*

If $f(x)$ is an odd function then all $a_n = 0$ and therefore f has a sine-series.

Indeed, first of all this is intuitive, since sines are odd, while cosines are even. On the other hand, the equality (33) is on average, so we should better check using formulas. And it is true, since: if $f(x)$ is even, then all $f(x)\sin(nx)$ are odd, hence their integral on a symmetric interval is zero, hence $b_n = 0$; if $f(x)$ is odd, then all $f(x)\cos(nx)$ are odd, hence all $a_n = 0$.

Now if $f(x)$ is an arbitrary function on $[\pi, \pi]$ (or on any symmetric interval), it can be written as a sum of an even function, $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ (its *even part*) and an odd function $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$ (its *odd part*): $f = f_e + f_o$. Hence, if $f(x)$ is defined on $[-\pi, \pi]$ its Fourier series (33) equals the sine-series of its odd part f_o plus the cosine-series of its even part f_e.

4.2.2. *Functions on* $[0, \pi]$. Given any function $g(x)$ for $x \in [0, \pi]$, then g can be continued to $x \in [-\pi, \pi]$ by

1) requiring that the function on $[-\pi, \pi]$ be odd, that is continued as

$$g_{odd}(x) = \begin{cases} g(x) & \text{if } 0 < x < \pi \\ -g(-x) & \text{if } -\pi < x < 0 \end{cases}$$

in which case $g(x)$ has a sine-series, or by

2) requiring that the function on $[-\pi, \pi]$ be even, that is continued as

$$g_{even}(x) = \begin{cases} g(x) & \text{if } 0 < x < \pi \\ g(-x) & \text{if } -\pi < x < 0 \end{cases}$$

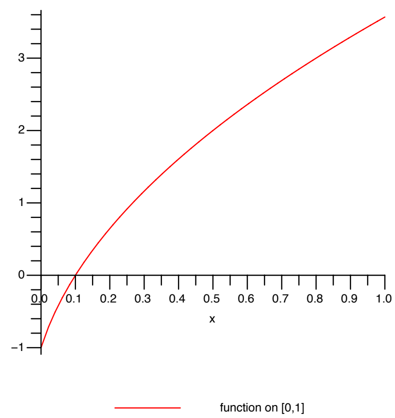
in which case $g(x)$ has a cosine-series.

(Recall that functions that differ by the value at one point, such as $x = 0$, are considered equal in L^2.)

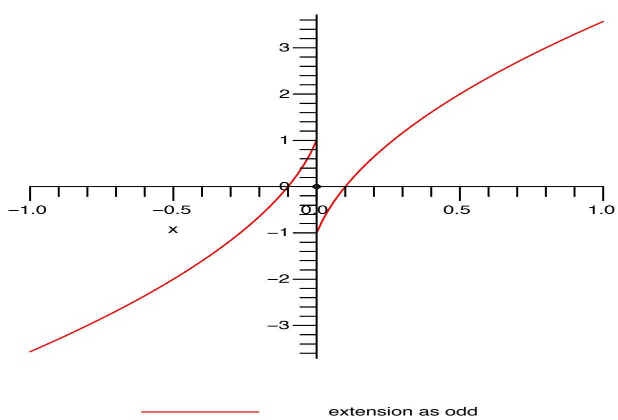
Note that the odd part of g_{odd} is g, hence its Fourier series contains only sines (and equals the sine series of g); the even part of g_{even} is also g, hence its Fourier series contains only cosines (and equals the cosine series of g).

We have now fully reconciled the Fourier series with the sine and cosine series.

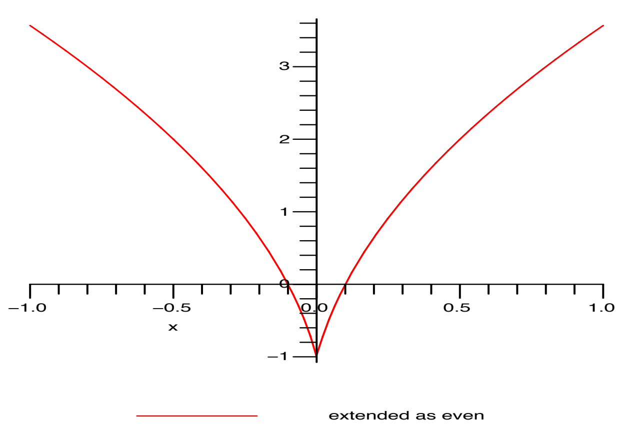
The figure below shows a function on $[0, 1]$



and its extension to $[-1, 1]$ as an odd function



and its extension to $[-1, 1]$ as an even function



4.3. **General intervals.** First note that instead of the interval $[-\pi, \pi]$ we can use any interval of length 2π (but we loose the odd-even symmetry); similarly, instead of the interval $[0, \pi]$ for the sine and cosine series, we can use any interval of length π. For a function $f(x)$ on a general interval $[a, b]$ a linear change of the coordinate x shows that we can write an expansion

$$f = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{2\pi i n x/(b-a)} \quad \text{ on } [a, b]$$

or, we can replace x by any translation $x - c$. Also sine series and cosine series ca be written, with argument $n\pi x/(b - a)$. The general Fourier series is more appropriate if f is periodic on $[a, b]$ (i.e. $f(a) = f(b)$).

4.4. **Point-wise convergence of Fourier series.** While the Fourierseries (32) and (33) converges to f in the L^2-norm (that is, in squared average), it is useful to know when the series converges point-wise (that is, for a fixed x, as a series of numbers):

Theorem 6. Point-wise convergence of a Fourier series

A. *If f and f' are piecewise continuous on $[-\pi, \pi]$ then the series (33) converges for every x.*

B. *Let $c \in (-\pi, \pi)$.*
(i) If f is continuous at $x = c$ then the Fourier series (33) at $x = c$ converges to $f(c)$.
(ii) If f has a jump discontinuity at $x = c$ then the Fourier series (33) at $x = c$ converges to $[f(c-) + f(c+)]/2$.

C. *The behavior of the Fourier series at the points $x = \pi$ and $x = -\pi$ is seen in the following way. Continue $f(x)$ outside $[-\pi, \pi]$ by 2π-periodicity. If $f(\pi-) = f(-\pi+)$ then $x = \pm\pi$ are points of continuity, and the series (33) converges to $f(\pm\pi)$ for $x = \pm\pi$. Otherwise, $x = \pm\pi$ are points where there is a jump discountinuity and the series (33) converges to $[f(\pi-) + f(-\pi+)]/2$ for $x = \pm\pi$.*

We will give a proof of Theorem 6 shortly.

Remark. *Smoothness of a function is related to the rate of decay of its Fourier coefficients.* Namely, the more derivatives a function has, the faster its Fourier coefficients decay.

Recall that if $f \in \mathcal{H}$ then sequence of its generalized Fourier coefficients with respect to an orthonormal basis of $\mathcal{H}$ belongs to ℓ^2. In particular, the sequence $a_0, a_1, b_1, a_2, b_2, \ldots$ in (33) is in ℓ^2, so $a_n, b_n \to 0$, faster than $1/\sqrt{n}$. It can be shown that if the coefficients decay faster, say like $1/n^2$, then f is continuous and piecewise differentiable. And so on: the faster the decay, the more derivatives f has.

Precise statements can be found in books dedicated to Fourier analysis.

4.5. **An example.** Let us find the sine-series of function $f(x) = 1$ for $x \in [0, \pi]$.

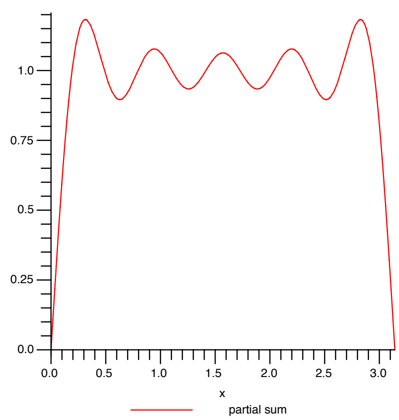
It can be checked that $b_n = 0$ for n even, and its sine-series is

$$\frac{4}{\pi}\sin(x) + \frac{4}{3\pi}\sin(3\,x) + \frac{4}{5\pi}\sin(5\,x) + \frac{4}{7\pi}\sin(7\,x) + \frac{4}{9\pi}\sin(9\,x) + \dots$$

For each $x \in (0, \pi)$ the sine-series converges to 1 by Theorem 6 **B.**(i). To understand the value at which the series converges at the end points $x = 0$ and $x = \pi$ we first continue the function to an odd function on $[-\pi, \pi]$, which is $f_{odd}(x) = 1$ for $x \in (0, \pi]$ and $f_{odd}(x) = -1$ for $x \in [-\pi, 0)$. Using Theorem 6 **B.**(i) the series at $x = 0$ converges to $[f_{odd}(0-) + f_{odd}(0+)]/2 = 0$ and by Theorem 6 **C.** at $x = \pi$ the series converges to $[f_{odd}(-\pi+) + f_{odd}(\pi-)]/2 = 0$.

Of course, by substituting directly $x = 0$ or $x = \pi$ in the series we see that all its terms are zero.

The picture shows the plot of the sum of the first five nonzero terms.



Note the large overshoot of the partial sum at the at the jump discontinuities at $x = 0$ and $x = \pi$: this is called Gibbs phenomenon. This behavior of truncates of Fourier series gives rise to artifacts in signal processing.

4.6. **Dirichlet Kernel.** [See the handwritten notes.]

4.7. **Proof of Theorem** (6)**.** [See the handwritten notes.]

4.8. **More about Gibbs phenomenon.** [See the hand-out on Carmen; handwritten notes may also come.]