The orthohmonic:

\[ L^2([-\pi, \pi]) \xrightarrow{\mathcal{F}} \ell^2 \]
\[ f \mapsto \{ \hat{f}_n \}_{n \in \mathbb{Z}} \]

\[ \mathcal{F}(f) = \hat{f} = \{ \hat{f}_n \}_{n \in \mathbb{Z}} \quad \text{where} \quad \hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} f(x) \, dx \]

Conversely:

\[ L^2([-\pi, \pi]) \xleftarrow{\mathcal{F}^{-1}} \ell^2 \]
\[ f \mapsto \{ \hat{f}_n \}_{n \in \mathbb{Z}} \]

\[ f = \mathcal{F}^{-1}(\{ \hat{f}_n \}) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx} \]

Moreover, we showed:

\[ \mathcal{F}(f \ast g) = \mathcal{F}(f) \cdot \mathcal{F}(g) \]

where the product of 2 sequences is understood entry-by-entry.
Sampling - the mathematician's perspective

Suppose we have a function \( f : [0, b] \to \mathbb{R} \)
and we know its values at sample points
\( a \leq x_0 < x_1 < \ldots < x_n \leq b \quad f(x) = \text{known} \) (unusually \( x = \text{limit} \))

Reconstruct \( f \) from the samples \( f(x_0), f(x_1), \ldots, f(x_n) \).

Of course, this is not possible! There are infinitely

many functions with the

prescribed values \( f(x) \) if \( b - a = n \).

What we can do, is find functions within given classes.

...we want to interpolate

Say \( [a, b] = [-\pi, \pi] \) (otherwise we need a linear change

of variables)

Sampling - 2N+1 points, equally spaced:
\[
x_k = \frac{2\pi k}{2N+1} = \frac{\pi}{N+\frac{1}{2}} \quad k = 0, 1, \ldots, N
\]

Given the values \( c_k = f\left(\frac{k\pi}{N}\right) \) for \( |k| \leq N \) find an

interpolating function. In what class?

- polynomials: and if a polynomial of \( \deg \leq 2N \) there is
  one unique!
- Fourier series. If there we look for a

in a subspace of dimension 2N+1 (trig polynomial)

there exists a unique \( f \)

Construct it: We search for a periodic, period 2\( \pi \)

with \( \hat{f}_n = 0 \) for \( |n| > N \), so basically we look for \( f \) a trig poly

\( f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \)

\[ \sum_{n=-N}^{N} \int x \sin x \]

Consider the normalized Dirichlet polynomials

\[ U_k(x) = \frac{1}{2^{N+1}} D_N \left( x - \frac{2\pi k}{2N+1} \right) \] \( \text{for } k = 0, 1, 2, \ldots, 2N \)

Note its values on our samples \( x_e = \frac{2\pi e}{2N+1} \):

\[ U_k(x_e) = 0 \text{ if } k \neq l \text{ since } \]

\[ D_N \left( x - \frac{2\pi k}{2N+1} \right) = \frac{\sin \left( (N+\frac{1}{2}) \left( x - \frac{2\pi k}{2N+1} \right) \right)}{\sin \left( \frac{2\pi k}{2N+1} \right) \frac{1}{2}} = 0 \]

and

\[ U_k(x_e) = \frac{1}{2^{N+1}} D_N (0) = \frac{1}{2^{N+1}} \sum_{n=-N}^{N} e^{2\pi i nx} = 1 \]

(this is why we want the normalization \( \frac{1}{2N+1} \))

Therefore the function

\[ g(x) = \sum_{k=-N}^{N} \hat{f}(x_k) U_k(x) \]

has the same values as \( f \) at the sample points.
\[ g(x) = \sum_{k=-N}^{N} f(x_k) \delta(x-k) = \sum_{k=-N}^{N} f(x_k) \delta_{x_k} = f(x) \]

So \( g \) is an interpolation of the values \( f(x_k), k=-N, \ldots, N \) and \( g \) is a trig. polyn where only \( \cos(kx), \sin(kx) \) with \( 0 \leq k \leq N \) appear, and no other frequencies. On the other hand, this is the only trig. polyn with these frequencies. So, here is the "reconstructed" signal.

\[ \hat{f}(x) = \sum_{k=-N}^{N} f(x_k) \frac{D_n}{2N+1} D_n(x - \frac{2\pi k}{2N+1}) \]

is the unique function which has the same samples as \( f \) and Fourier components \( \hat{f}_n \) nonzero only for \( |n| \leq N \). This is the compact interval revision of Shannon's theorem (or Whittaker-Shannon Theorem).

Note: We also showed this way that the functions \( U_k(x) = \frac{1}{2N+1} D_n(x - \frac{2\pi k}{2N+1}) \), \( k = 0, 1, \ldots, 2N \)

form a basis for the trig. polyn \( \mathbb{S}_{N+1}, \cos(kx), \sin(kx) | k = 0, 1, \ldots, N \)

and \( \mathbb{S}_{N+1} e^{ikx} | |k| \leq N \)

It is precisely the basis that can exactly be used for "reconstruction" from samples. If it really is only interpolation.