The situation:

$$L^2[-\pi,\pi] \xrightarrow{\mathcal{F}} \ell^2$$
$$f \longmapsto \{\hat{f}_n\}_{n\in\mathbb{Z}}$$

$\mathcal{F}(f) = \{\hat{f}_n\}_{n\in\mathbb{Z}}$ where $\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} f(x)\,dx$

Conversely

$$L^2(-\pi,\pi) \xleftarrow{\mathcal{F}^{-1}} \ell^2$$
$$f \longleftarrow \{\hat{f}_n\}_{n\in\mathbb{Z}}$$
$$f = \mathcal{F}^{-1}(\{\hat{f}_n\}_n) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$$

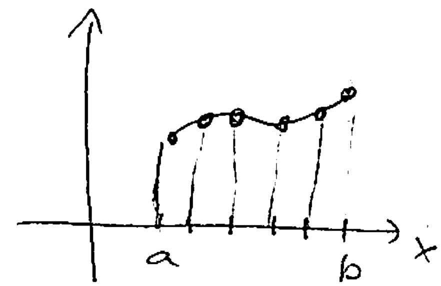
Moreover, we showed:

$$\mathcal{F}(f * g) = \mathcal{F}(f)\cdot\mathcal{F}(g)$$

where the product of 2 sequences is understood - entry-by-entry

Sampling – the mathematician's perspective

Suppose we have a function $f : [a,b] \to \mathbb{R}$
and we know its values at sample points
$a \le x_0 < x_1 < \ldots < x_n \le b$ $\quad f(x_k) =$ known. (usually $x =$ time)
Reconstruct f from the samples $f(x_0), f(x_1) \ldots f(x_n)$



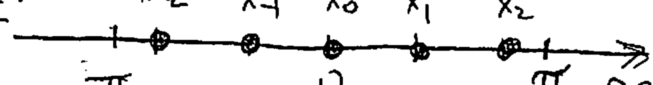
Of course, this is not possible! There are infinitely many functions with the prescribed values $f(x_k)$, $k = 0 \ldots n$.

What we can do, is find functions within given classes.
– we want to interpolate

Say $[a,b] = [-\pi, \pi]$ (otherwise we need a linear change of variables)

Sampling: $2N+1$ points, equally spaced: $x_k = \frac{2\pi k}{2N+1} = \frac{\pi}{N+\frac{1}{2}} k$
$|k| \le N$

$\underline{N=2}$:



Note: the endpoints are not sampled here, so that we can require periodicity

Given the values $c_k = f\left(\frac{k\pi}{N}\right)$ for $|k| \le N$ find an interpolating function. In what class?
– polynomials: and if polyn of deg $\le 2N$ there is one unique!
or
– splines
or

- Fourier series. If there we look for f in a subspace of dimension $2N+1$ (trig polynomial) there exists a unique f

Construct it: We search for f periodic, period 2π with $\hat{f}_n = 0$ for $|n| > N$, so basically we look for f a trig polyn $f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$

$$= \sum_{n=-N}^{N} \hat{f}_n e^{inx}$$

Consider the normalized Dirichet polynomials

$$u_k(x) = \frac{1}{2N+1} D_N\left(x - \frac{2\pi k}{2N+1}\right) \text{ for } k = 0, 1, 2 \ldots 2N$$

Note its values on our samples $x_\ell = \frac{2\pi \ell}{2N+1}$:

$u_k(x_\ell) = 0$ if $k \neq \ell$ since

$$D_N\left(x_\ell - \frac{2\pi k}{2N+1}\right) = \frac{\sin(N+\frac{1}{2})(x_\ell - \frac{2\pi k}{2N+1})}{\sin(x_\ell - \frac{2\pi k}{2n+1})\frac{1}{2}} = \frac{\sin \frac{2N+1}{2} \frac{2\pi(\ell-k)}{2N+1}}{*} = 0$$

and

$$u_k(x_k) = \frac{1}{2N+1} D_N(0) = \frac{1}{2N+1} \sum_{n=-N}^{N} e^{i0x} = 1$$

(this is why we want the normalization $\frac{1}{2N+1}$)

Therefore the function

$$g(x) = \sum_{k=-N}^{N} f(x_k) u_k(x)$$

has the same values as f at the sample points.

$$g(x_e) = \sum_{k=-N}^{N} f(x_e)\, u_e(x) = \sum_{k=-N}^{N} f(x_e)\,\delta_{ke} = f(x_k)$$

So g is an intepolation of the values $f(x_k), k=-N..N$ and g is a trig. polyn where only $\cos kx$, $\sin kx$, with $0 \le k \le N$ appear, and no other frequencies. On the other hand, this is the only trig polyn with these frequencies so, here is the "reconstructed" signal.

$$"f(x)" = \sum_{k=-N}^{N} f(x_k) \frac{1}{2N+1} D_N\left(x - \frac{2\pi k}{2N+1}\right).$$

is the unique function which has the same samples as f and Fourier components $\hat{f}_n$ nonzero only for $|n| \le N$.

This is the compact interval version of Shannon's theorem

(or, Whittaker-Shannon Theorem)

Note: We also showed this way that the functions $u_k(x) = \frac{1}{2N+1} D_N\left(x - \frac{2\pi k}{2N+1}\right)$, $k = 0, 1, .. 2N$ form a 'basis' for the trig polyn

$$Sp\{1, \cos kx, \sin kx \mid k = 0, 1..N\}$$

and $Sp\{e^{ikx} \mid |k| \le N\}$

It is precisely the basis that can easily be used for "reconstruction" from samples.
it really is only interpolation.