Monday, Feb. 25, 2013

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O.N. Wave Packets

Key ideas

Completeness ppty of $\{P_{ij}\}$

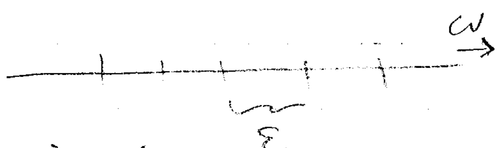
O.N. " " $\{P_{ij}\}$

Degrees of freedom $\|f\|^2_{L^2}$ = energy

phase space

phase space cells

Signal rep'n in phase space

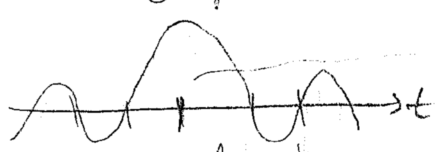


$\Delta\omega = \varepsilon$

$$F_{j\ell}(\omega) = \begin{cases} 0 & \text{otherwise} \\ \dfrac{e^{-2\pi i \ell \omega/\varepsilon}}{\sqrt{\varepsilon}} & j\varepsilon \le \omega \le (j+1)\varepsilon \end{cases}$$

$$P_{j\ell}(t) = \int_{-\infty}^{\infty} F_{j\ell}(\omega) \frac{e^{i\omega t}}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi\varepsilon}}\, e^{i(t - \frac{2\pi\ell}{\varepsilon})} \cdot \times \frac{\sin(t - \frac{2\pi\ell}{\varepsilon})\frac{\varepsilon}{2}}{t - \frac{2\pi\ell}{\varepsilon}}$$

equally spaced zeros:



Re $P_{j\ell}(t)$

Localized in t!

$t_\ell = \frac{2\pi\ell}{\varepsilon}$ $\qquad \Delta t = \frac{2\pi}{\varepsilon}$

Note $\mathcal{F}(e^{i\omega' t}) = \sqrt{2\pi}\,\delta(\omega - \omega')$

Fourier Integral Rep'n + Fourier Series Rep'n $\Rightarrow$
O.N. wavepacket rep'n.

We know that the set of wavepackets

$$\left\{P_{j\ell}(t)=\int_{-\infty}^{\infty}F_{j\ell}(\omega)e^{i\omega t}d\omega;\ j,\ell=\dots,\pm1,0,\pm1,\dots\right\}$$

for an orthonormal set:

$$\langle P_{j\ell}, P_{j'\ell'}\rangle=\delta_{jj'}\,\delta_{\ell\ell'}\,.$$

Q: But do these packet form a complete set, i.e. is it true that any $f(t)\in L^2(-\infty,\infty)$ can be represented by means of a series

$$f(t)=\sum_j\sum_\ell \alpha_{j\ell}\,P_{j\ell}(t)$$

for an appropriately chosen set of coefficients $\{\alpha_{ij}\}$?

A: The Fourier Integral Theorem together with the Fourier Series Theorem not only tell that the answer is "yes", but also

tell us what the α_{ij}-coefficient are.

The validation of these claims is a three step process.

1. Start with the Fourier spectral representation,

$$f(t)=\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\sqrt{2\pi}}\hat f(\omega)\,d\omega,$$

which is based on the Fourier Integral Theorem.

Break up this rep'n into its Fourier window subintervals:

$$f(t)=\dots\int_{(j-1)\epsilon}^{j\epsilon}\frac{e^{i\omega t}}{\sqrt{2\pi}}\hat f(\omega)\,d\omega+\int_{j\epsilon}^{(j+1)\epsilon}\frac{e^{i\omega t}}{\sqrt{2\pi}}\hat f(\omega)\,d\omega+\dots$$

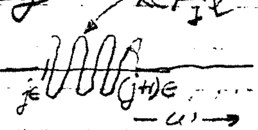
$$=\sum_{j=-\infty}^{\infty}\int_{j\epsilon}^{(j+1)\epsilon}d\omega\frac{e^{i\omega t}}{\sqrt{2\pi}}\hat f(\omega)$$

2. Apply the Fourier series theorem to $\hat f(\omega)$ on each subinterval $[j\epsilon,(j+1)\epsilon]$ and obtain

$$f(t) = \sum_j \underbrace{\int_{j\epsilon}^{(j+1)\epsilon} d\omega \frac{e^{i\omega t}}{\sqrt{2\pi}} \sum_{\ell=-\infty}^{\infty} \frac{e^{-2\pi i \ell \frac{\omega}{\epsilon}}}{\sqrt{\epsilon}}}_{P_{j\ell}(t)} \underbrace{\int_{j\epsilon}^{(j+1)\epsilon} \frac{e^{\frac{2\pi i \ell}{\epsilon}\omega'}}{\sqrt{\epsilon}} \hat{f}(\omega')\, d\omega'}_{\equiv\ \alpha_{j\ell}}$$

The Forier coefficient is given by

$$\alpha_{j\ell} = \int_{-\infty}^{\infty} \overline{F}_{j\ell}(\omega')\, \hat{f}(\omega')\, d\omega'$$
$$= \langle F_{j\ell}, \hat{f} \rangle$$



$\mathrm{Re}\,F_{j\ell}$

$j\epsilon$ $(j+1)\epsilon$ $\omega' \rightarrow$

where $F_{j\ell}(\omega)$ is the ℓ^{th} Fourier Series basis function whose support is confined to $[j\epsilon, (j+1)\epsilon]$. We therefore have

$$\alpha_{j\ell} = \langle F_{j\ell}, \hat{f} \rangle$$
$$= \langle \hat{P}_{j\ell}, \hat{f} \rangle$$
$$= \langle P_{j\ell}, f \rangle \quad \text{Parseval's identity}$$

$$P_{j\ell}(t) = \int_{-\infty}^{\infty} \underbrace{F_{j\ell}(\omega)}_{\equiv \hat{P}(\omega')} \frac{e^{i\omega t}}{\sqrt{2\pi}}\, d\omega'$$

Consequently,

$$f(t) = \sum\sum P_{j\ell}(t)\, \alpha_{ij}; \quad \alpha_{ij} = \int_{-\infty}^{\infty} \overline{P}_{j\ell}(t') f(t')\, dt'$$

3. Introduce the definition of $P_{j\ell}(t)$, namely

$$P_{j\ell}(t) = \int_{-\infty}^{\infty} F_{j\ell}(\omega') \frac{e^{i\omega' t}}{\sqrt{2\pi}}\, d\omega'$$

so that

$$\hat{P}_{j\ell}(\omega') = F_{j\ell}(\omega')$$

Using Parseval's identity

$$\langle \hat{P}_{j\ell}, \hat{f} \rangle = \langle P_{j\ell}, f \rangle$$

one obtain

$$\alpha_{j\ell} = \langle F_{j\ell}, \hat{f} \rangle = \langle P_{j\ell}, f \rangle$$

Consequently

$$\boxed{\begin{aligned} f(t) &= \sum\sum \alpha_{j\ell}\, P_{j\ell}(t) \\ \alpha_{j\ell} &= \int_{-\infty}^{\infty} \overline{P_{j\ell}}(t')\, f(t')\, dt \end{aligned}}$$

and upon taking the Fourier xform, we also obtains

$$\boxed{\hat{f}(\omega) = \sum\sum \alpha_{j\ell}\, \hat{P}_{j\ell}(\omega)}$$

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Thus one has the following theorem

Given: 1.) $f \in L^2(-\infty,\infty)$

2.) $\left\{F_{j\ell}(\omega)\right\} = \left\{\frac{e^{-2\pi i \ell \omega/\epsilon}}{\sqrt{\epsilon}}\right\}_{\ell=-\infty}^{\ell=\infty}$

on each frequency window $j\epsilon \leq \omega \leq (j+1)\epsilon$, $j = 0, \pm 1, \ldots$

Conclusion:

The set of wave packets

$$P_{j\ell}(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\sqrt{2\pi}} F_{j\ell}(\omega)\, d\omega$$

$$= \sqrt{\frac{2}{\pi\epsilon}}\; e^{i\left(t - \frac{2\pi\ell}{\epsilon}\right)(j+\frac{1}{2})\epsilon}\; \frac{\sin\left(t - \frac{2\pi\ell}{\epsilon}\right)\frac{\epsilon}{2}}{t - \frac{2\pi\ell}{\epsilon}}$$

$j, \ell = 0, \pm 1, \pm 2, \ldots$

is a complete set of o.n. of ~~basis~~ fns for $L^2(-\infty,\infty)$.

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Corollary:

$$\sum\sum P_{j\ell}(t)\, \overline{P_{j\ell}(t')} = \delta(t-t') \qquad (*)$$

Proof: From page 4 one has

$$f(t) = \sum_j \sum_\ell P_{j\ell}(t) \int_{-\infty}^{\infty} \overline{P_{j\ell}(t')}\, f(t')\, dt'$$

Interchange the summation and the integr yields

$$f(t) = \int_{-\infty}^{\infty} \sum\sum P_{j\ell}(t)\, \overline{P_{j\ell}(t')}\, f(t')\, dt$$

QED.

Note that, upon multiplying both sides of Eq $(*)$ at the top of this page by $f(t')$ and integrating, one obtains

$$\sum\sum P_{j\ell}(t)\, \alpha_{ij} = f(t).$$

Thus Eq $(*)$ is a restatement of the fact that $\{P_{j\ell}\}$ is a spanning set for $L^2_{(-\infty,\infty)}$.

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i.e. $\{P_{je}\}$ is a complete set of basis functions.

Degrees of Freedom

Relative to $\{P_{j\ell}\}$, an o.n. basis for $L^2(-\infty,\infty)$, a function is identified by specifying the complex parameters α_{ij}.

$$f(t) = \sum\sum \alpha_{ij} P_{ij}(t)$$
$$\hat{f}(\omega) = \sum\sum \alpha_{ij} \hat{P}_{ij}(\omega)$$

In light of the orthonormality of these basis fns, one finds that the total energy of the signal function is

$$\langle f, f\rangle = \langle \sum\sum \overline{\alpha_{ij}}\, \overline{P_{ij}}, \sum\sum \alpha_{i'j'} P_{i'j'} \rangle$$
$$= \sum\sum |\alpha_{ij}|^2 = \langle \hat{f}, \hat{f}\rangle$$

Thus each $|\alpha_{ij}|^2$ is the energy associated with the $(ij)^{th}$ component

$$\alpha_{ij} P_{ij}(t)$$

of the signal $f(t)$ synthesized in terms the wave packets $\{P_{j\ell}\}$. These wave packet constitute an isometric mapping between square summable series and square integrable functions

$$\ell^2 \longleftrightarrow L^2(-\infty,\infty)$$
$$\alpha_{ij} \quad \{P_{ij}\}$$
$$\{\alpha_{ij}\} \rightsquigarrow f(t)$$

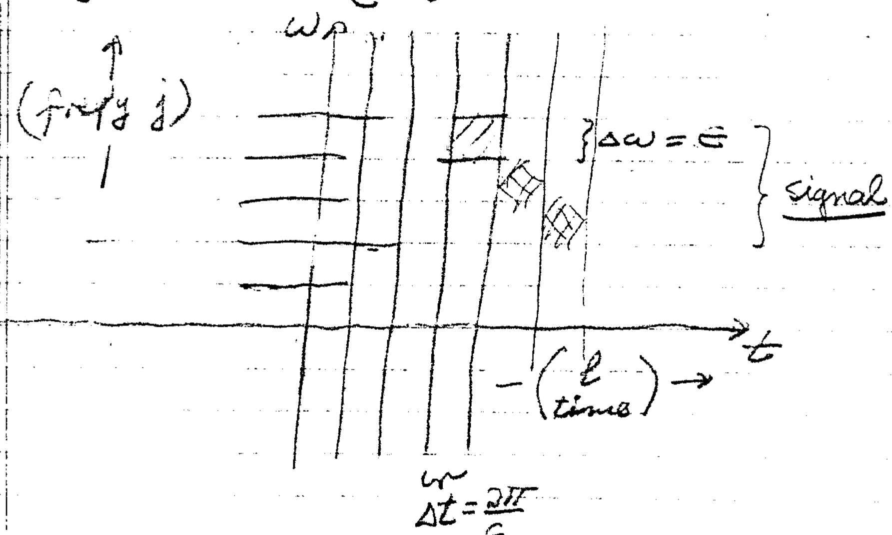
In physics and engineering each complex $\alpha_{ij} \in \mathbb{C}$ is the $(ij)^{th}$ degree of freedom for the system of signals $f \in L^2$. By varying this complex parameter

The phase space rep'n of

$$f = \sum\sum \alpha_{ij} P_{ij}(t)$$

$$\hat{f} = \sum\sum \alpha_{ij} \hat{P}_{ij}(\omega)$$

consists of the domain spanned by t and ω, but each one partitioned into discrete subintervals of size $\Delta t = \frac{2\pi}{\epsilon}$, $\Delta\omega = \epsilon$.



$$\begin{pmatrix}\text{area of phase}\\ \text{space cell}\end{pmatrix} = \Delta t \Delta\omega = \frac{2\pi}{\epsilon}\epsilon = 2\pi \quad \forall (i,j) \text{ cells}$$

(incompressible, i.e. independ. of ϵ)

$$\langle f, f\rangle = \sum\sum |\alpha_{ij}|^2$$

$$\langle \hat{f}, \hat{f}\rangle = \sum\sum |\alpha_{ij}|^2$$

$|\alpha_{ij}|^2$ = energy in $(i,j)^{th}$ degree of freedom

= " " $(i,j)^{th}$ phase space cell

There are exactly as many phase space cells as there are degrees of freedom.

Q: How many degrees of freedom are there in a system of bandlimited functions with support

$$(j-\tfrac{1}{2})\epsilon \qquad (j+\tfrac{1}{2})\epsilon$$

$$\underbrace{-j_{max}\epsilon = -\Omega \qquad\qquad \Omega = j_{max}\epsilon}_{\Delta\Omega}$$

and duration

$$\underbrace{0 \qquad\qquad T}_{\Delta T} = l_{max}\frac{2\pi}{\epsilon}$$

A:

$$\frac{\Delta\Omega}{\epsilon} = 2\left(j_{max}+\tfrac{1}{2}\right)\epsilon = (2j_{max}+1)$$

$$\frac{\Delta T}{\frac{2\pi}{\epsilon}} = l_{max}$$

# of degrees of freedom $= (2j_{max}+1) \times l_{max}$

$= \dfrac{\text{accessible amount of phase space area}}{\text{area / phase space cell}}$

$$= \frac{\Delta\Omega \cdot \Delta T}{\epsilon \cdot \frac{2\pi}{\epsilon}} = \frac{2\Omega \cdot T}{2\pi} =$$