Homework 10

1. The following problem arises in describing an axially symmetric wave propagating in a cylindrical pipe of radius a:

   Consider the eigenvalue problem

$$-\frac{d}{dr}\, r\, \frac{du}{dr} = k^2 r u \qquad\qquad 0 \le r \le a$$
$$u(0) = \text{finite}$$
$$u(a) = 0.$$

   The eigenfunctions are $u_m(r) = J_0(rk_m)$ where the boundary condition $J_0(ak_m) = 0$ determines the eigenvalues $k_m^2 \quad m = 1, 2, \ldots$.

   a) SHOW that $\{J_0(rk_m)\}$ is an orthogonal set of eigenfunctions on $(0, a)$.

   b) Using Exercise 3.3.5 ("How to normalize an eigenfunction") in Ch 3 of the text, FIND the squared norm of $J_0(rk_m)$.

   c) EXHIBIT the set of orthonormal eigenfunctions.

   d) FIND the Green's function for the above boundary value problem. (To get full credit this function should be expressed in closed form, not as an infinite series.)

2. PROVE that

$$N_{-n}(\rho) = (-1)^n N_n(\rho)$$

3. On a circular disc of radius a, FIND an orthonormal set of eigenfunctions for the system defined by the eigenvalue problem

$$-\nabla^2 \psi = k^2 \psi$$
$$\frac{\partial \psi}{\partial r}(r = a, \theta) = 0 \qquad\qquad \text{a = radius of disc}$$
$$\psi(r = 0, \theta) = \text{ finite} \qquad\qquad 0 \le \theta \le 2\pi$$

   where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$, and EXIBIT these eigenfunctions in their optimally simple form.

4. Consider a wave disturbance ψ which is governed by the wave equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right]\psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}.$$

Let this wave *propagate* inside an infinitely long cylinder; in other words, it satisfies

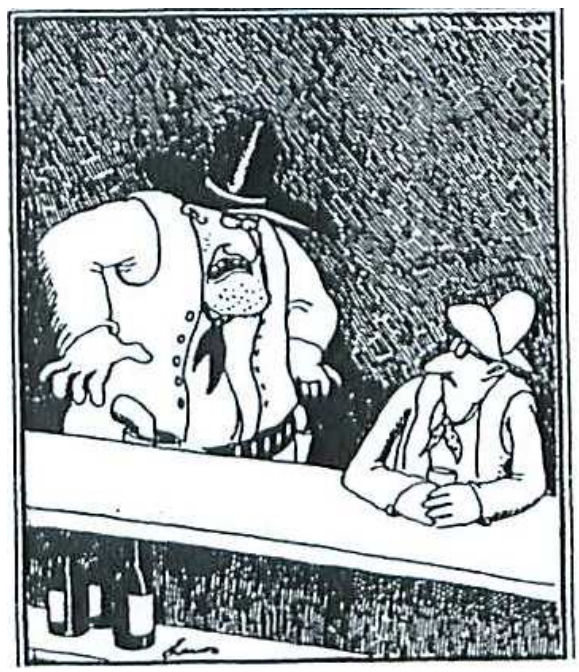
$$\frac{\partial \psi}{\partial z} = ik_z \psi$$

where k_z is some *real* number, not equal to zero. Assume that the boundary conditions satisfied by ψ is

$$\psi(r = a) = 0 \quad \text{with } a = \quad \text{radius of cylinder}$$

$$\psi(r = 0) = \quad \text{finite}$$

EXHIBIT a) the amplitude profiles, b) their dispersion relations, and c) FIND the "cut off" frequency, i.e. that frequency $\omega = \omega_{\text{critical}}$ below which *no propagation* in the infinite cylinder *is possible* at all.



"I asked you a question, buddy ...
What's the critical frequency, ω_{critical}, in terms of a and c to an accuracy of 2% or better?"

Also, ANSWER the question posed by the gentleman.