1. For each of the matrices $A$ below:
   a) classify $A$ (self-adjoint, unitary etc.);
   b) find a unitary $U$ which diagonalizes $A$;
   c) give the spectral decomposition in terms of the eigenspace projections;
   d) verify the resolution of the identity (completeness relation)

   (i) $A = \begin{bmatrix} 2 & i & 0 \\
   -i & 1 & -i \\
   0 & i & 2 \end{bmatrix}, \quad i = \sqrt{-1}$;
   (ii) $A = \begin{bmatrix} 1 & 1 & -1 \\
   1 & 1 & 1 \\
   -1 & 1 & 1 \end{bmatrix}$

2. Show that the following set of functions $V$ is a complex vector space:

   $V = \{ f : [0,1] \to \mathbb{C} | f \text{continuous} \}$

   (i) Show that $\langle f, g \rangle_1 = \int_0^1 fg \, dx$ is an inner product on $V$.
   Show that the integration operator $Tf(x) = \int_0^x f(t) \, dt$ is linear on $V$ and
   find its adjoint $T^*$.

   (ii) Show that $\langle f, g \rangle_w = \int_0^1 wfg \, dx$ is an inner product on $V$ if $w$ is a
   positive function ($w$ is called a weight).
   Show that the integration operator $Tf(x) = \frac{1}{w(x)} \int_0^x f(t) \, dt$ is linear on $V$
   and find its adjoint $T^*$ with respect to the inner product $\langle \cdot, \cdot \rangle_w$.

   Warning: these are not a Hilbert spaces (they are not complete in these norms).

3. Suppose $(e_j)_{j=1\ldots n}$ and $(f_j)_{j=1\ldots n}$ are orthonormal bases for $\mathbb{C}^n$. Sup-
   pose $M : \mathbb{C}^n \to \mathbb{C}^n$ is the linear operator satisfying $Me_j = f_j$.
   Show that $M$ is unitary.

4. Prove that no matrix $\begin{bmatrix} 1 & c \\
   0 & 1 \end{bmatrix}$ is similar to a real or complex diagonal
   matrix if $c \neq 0$. Interpret the result geometrically in terms of a transform-
   ation of points in $R^2$ onto points in $R^2$. 

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5.a) If $A$ has eigenvalues 0 and 1, corresponding to the eigenvectors \[
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\quad \text{and} \quad \begin{bmatrix}
2 \\
-1
\end{bmatrix},
\] how can one tell in advance that $A$ is self-adjoint and real?

b) What is its trace? Its determinant?

c) What is $A$?

6. a) (Reminder)

a. Show that the $n$ complex solutions of the equation $z^n = 1$ are $z_k = \exp(2\pi ik/n)$, $k = 0, 1, \ldots, n - 1$ (they are called the $n^{th}$ roots of unity).

b. Show that $z_k = z_1^k$. ($z_1$ is called the “primitive” $n^{th}$ root of unity.)

c. Why do we stop the index at $k = n - 1$? What is $z_n, z_{n+1}$?

d. Plot these $n$ solutions $z_0, z_1, \ldots, z_{n-1}$ in the complex plane for (i) $n = 2$; (ii) $n = 3$; (iii) $n = 4$; (iv) for a general $n$. (Use the one separate plane for each $n$).

b) Let $w = e^{2\pi i/n}$ be the “primitive” $n^{th}$ root of unity.

Consider $[U_{jk}] = [w^{jk}] \frac{1}{\sqrt{n}}$, $j, k = 0, 1, \ldots, n - 1$, which is the Fourier Matrix $U$:

\[
U = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & w & w^2 & \cdots & w^{n-1} \\
1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2}
\end{bmatrix} \frac{1}{\sqrt{n}} = [U_{jk}].
\]

TRUE or FALSE: $U$ is unitary. Why?

7. Define $\exp(A)$, where $A$ is any square matrix, by the power series expansion of $\exp(x) = 1 + x + \frac{1}{2}x^2 + \cdots$ (Assume that the series $\exp(A)$ does converge... it does.)

Show that if $A$ is diagonalizable, i.e. there exists a matrix $S$ which reduces $A$ to diagonal form by a similarity transform, then one has

$$
\det \exp(A) = \exp \tr(A).
$$