Math 602 **Homework 7**

The problems here are well known results. You need to show all your work and explanations.

1. (i) Check that the functions 1, $\sin(nx)$, $\cos(nx)$, $(n = 1, 2, 3\ldots)$ form an orthogonal system in $L^2[-\pi, \pi]$.

(ii) Normalize them to obtain an orthonormal system.

(iii) Assuming that $f \in L^2[-\pi, \pi]$ has its Fourier series expansion

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{1}$$

verify that the Fourier coefficients a_n and b_n are given by the formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, \cos(nx)\, dx\ , \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, \sin(nx)\, dx,$$

2. (i) Check that the functions e^{inx}, $n \in \mathbb{Z}$ form an orthogonal system in the complex valued functions $L^2[-\pi, \pi]$.

(ii) Normalize them to obtain an orthonormal system.

(iii) Assuming that $f \in L^2[-\pi, \pi]$ has its Fourier series

$$f = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx} \tag{2}$$

verify that the Fourier coefficients are given by

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

(iv) Verify that the Fourier coefficients $a_n, b_n, \hat{f}_n$ of a function $f(x)$ are related by the formulas

$$a_n = \hat{f}_n + \hat{f}_{-n} \text{ for } n = 0, 1, 2, \ldots, \quad b_n = i(\hat{f}_n - \hat{f}_{-n}) \text{ for } n = 1, 2, \ldots$$

(v) Find the conditions on $\hat{f}_n$ that ensure that the function $f(x)$ is real valued.

4. Use a change of the variable x in (2) to show that the Fourier series of a function $g \in L^2[a,b]$ has the form

$$g = \sum_{n=-\infty}^{\infty} \hat{g}_n e^{2\pi i n x/(b-a)}$$

and find the formula that expresses $\hat{g}_n$ in terms of $g(x)$.

5. The Legendre orthogonal polynomials are orthogonal in $L^2[-1,1]$.

Use a Gram-Schmidt process on the polynomials $1, x, x^2, x^3$ to obtain an orthonormal set; these are the first four Legendre polynomials.

6. The Laguerre orthogonal polynomials are orthogonal in the weighted $L^2([0,+\infty), e^{-x}dx)$.

Use a Gram-Schmidt process on the polynomials $1, x, x^2, x^3$ to obtain an orthonormal set; these are the first four Laguerre polynomials.