Please note the following very useful inequalities:

\[ |f(t)| \leq \|f\|_{\infty} \quad \text{for all } t \in S \text{ if } \|f\|_{\infty} = \sup_{x \in S} |f(x)| \]

and

\[ |\int_a^b f(t) \, dt| \leq \int_a^b |f(t)| \, dt \]

1. Consider the sequence of functions \( f_n(x) = x^n \).
   a) Show that the sequence \( f_n \) is point-wise convergent for all \( x \in [0,1] \), that is, show that for all \( x \in [0,1] \) the limit \( \lim_{n \to \infty} f_n(x) = f(x) \) exists; what is the limit function \( f(x) \)?
   b) The functions \( f_n \) belong to the space \( C[0,1] \) of continuous functions on \([0,1]\). Do they converge in the sup norm in this space?
   c) The functions \( f_n \) belong to the space \( L^2[0,1] \). Do they converge in the \( L^2 \) norm in this space?

2. Consider the integral operator \( I : C[0,2] \to C[0,2] \) given by
   \[ I(f) = \int_0^x f(t) \, dt \]
   a) Show that \( I \) is a linear transformation.
   b) Show that \( I \) is a bounded operator, that is, there exists a number \( B \) so that
   \[ \|I(f)\|_{\infty} \leq B\|f\|_{\infty} \quad \text{for all } f \in C[0,2] \]
   and find the (smallest) number \( B \).

3. Consider the eigenvalue problem (separation of variables in the equation of a vibrating string)
   \[ (T(x)y'(x))' + \lambda \rho(x)y(x) = 0, \quad x \in [0,L], \quad \text{where } T(x), \rho(x) > 0 \]
   with boundary conditions
   \[ y(0) = 0, \quad my'(L) + ky(L) = 0 \]
   (the endpoint \( x = 0 \) is kept fixed, and the other end, \( x = L \) is accelerated up and down with no transversal force).
   Reformulate this problem using a self-adjoint differential operator on an appropriate Hilbert space: give the operator, its domain, and show that it is self-adjoint. Is this operator positive definite?
   Without explicitly solving the problems, what can you say about the eigenvalues and eigenfunctions of this problem?