Math 602, 2012 **Homework 8**

Please note the following very useful inequalities:

$$|f(t)| \leq \|f\|_\infty \quad \text{for all } t \in S \text{ if } \|f\|_\infty = \sup_{x\in S} |f(x)|$$

and

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b \big|f(t)\big|\, dt$$

1. Consider the sequence of functions $f_n(x) = x^n$.

a) Show that the sequence f_n is point-wise convergent for all $x \in [0,1]$, that is, show that for all $x \in [0,1]$ the limit $\lim_{n\to\infty} f_n(x) = f(x)$ exists; what is the limit function $f(x)$?

b) The functions f_n belong to the space $C[0,1]$ of continuous functions on $[0,1]$. Do they converge in the sup norm in this space?

c) The functions f_n belong to the space $L^2[0,1]$. Do they converge in the L^2 norm in this space?

2. Consider the integral operator $I : C[0,2] \to C[0,2]$ given by

$$I(f) = \int_0^x f(t)\, dt$$

a) Show that I is a linear transformation.

b) Show that I is a bounded operator, that is, there exists a number B so that

$$\|I(f)\|_\infty \leq B\|f\|_\infty \quad \text{for all } f \in C[0,2]$$

and find the (smallest) number B.

3. Consider the eigenvalue problem (separation of variables in the equation of a vibrating string)

$$(T(x)y'(x))' + \lambda\rho(x)y(x) = 0, \quad x \in [0,L], \quad \text{where } T(x), \rho(x) > 0$$

with boundary conditions

$$y(0) = 0, \quad m\, y'(L) + k\, y(L) = 0$$

(the endpoint $x = 0$ is kept fixed, and the other end, $x = L$ is accelerated up and down with no transversal force).

Reformulate this problem using a self-adjoint differential operator on an appropriate Hilbert space: give the operator, its domain, and show that it is self-adjoint. Is this operator positive definite?

Without explicitly solving the problems, what can you say about the eigenvalues and eigenfunctions of this problem?