

Recall:  $\{e^{inx}\}_{n \in \mathbb{Z}}$  form an orthogonal basis

for  $L^2(-\pi, \pi)$ . That is, any  $f \in L^2(-\pi, \pi)$  can be uniquely represented as a Fourier series

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$$

and the sequence  $\{\hat{f}_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ , that is,  $\sum_{n=-\infty}^{\infty} |\hat{f}_n|^2 < \infty$

**Sin-Cos series:**

when  $f$  has real values  $f(x) \in \mathbb{R}$  we may prefer to write a series with real terms

[For simplicity here  $\hat{f}(-n) \equiv \hat{f}_n$ ]

$$\text{so } S(f(x)) = \hat{f}_0 + \sum_{n=1}^{\infty} (e^{inx} \hat{f}_n + e^{-inx} \hat{f}_n) \in \mathbb{R} \iff \hat{f}_n = \overline{\hat{f}_n}$$

So if  $\hat{f}_n = A_n + iB_n$  and  $\hat{f}_{-n} = A_n - iB_n$  then

$$Sf(x) = \hat{f}_0 + \sum_{n=1}^{\infty} 2 \operatorname{Re}(e^{inx} \hat{f}_n) = \hat{f}_0 + \sum_{n=1}^{\infty} 2(A_n \cos(nx) - B_n \sin(nx))$$

$$= \hat{f}_0 + \sum_{n=1}^{\infty} \left( \underbrace{2A_n}_{a_n} \cos(nx) - \underbrace{2B_n}_{b_n} \sin(nx) \right)$$

Denote  $a_n = 2A_n$ ,  $b_n = -2B_n$ .

$$\text{then } \underline{a_n} = \hat{f}_n + \hat{f}_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{inx} + e^{-inx}) f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x) dx$$

$$\text{and } \underline{b_n} = \frac{-1}{2} (\hat{f}_n - \hat{f}_{-n}) = \frac{-1}{2\pi i} \int_{-\pi}^{\pi} (e^{-inx} - e^{inx}) f(x) dx = \frac{1}{2\pi i} \int_{-\pi}^{\pi} (2i \sin(nx)) f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x) dx$$

$$\text{while } \underline{\hat{f}_0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2} \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(0x) f(x) dx = \frac{1}{2} a_0$$

for convenience!

So: if  $f(x)$  is real-valued,  $f: [-\pi, \pi] \rightarrow \mathbb{R}$   
 we write  $f \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$

$$\text{where } \hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) dx$$

and for  $f$  real then  $f(x) = \overline{f(x)} \Rightarrow \hat{f}_{-n} = \overline{\hat{f}_n}$

or we can write

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x) dx, \quad n=0, 1, 2, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x) dx, \quad n=1, 2, \dots$$

$$\text{and } a_n = 2 \operatorname{Re} \hat{f}_n$$

$$b_n = -2 \operatorname{Im} \hat{f}_n$$

The set of functions

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$$

form an orthogonal basis for  $L^2(-\pi, \pi)$

\* \* \*

In other intervals:  $f \in L^2(a, b) \rightarrow \mathbb{R}$

$$f \sim \sum \hat{f}_n e^{in \frac{2\pi x}{b-a}}, \quad \hat{f}_n = \frac{1}{b-a} \int_a^b e^{in \frac{2\pi x}{b-a}} f(x) dx$$

$$f \sim \frac{a_0}{2} + \sum a_n \cos\left(n \frac{2\pi x}{b-a}\right) + b_n \sin\left(n \frac{2\pi x}{b-a}\right)$$

Using either representations, we know that if  $f \in L^2(-\pi, \pi)$  then its Fourier series converges to  $f$  in mean square sense, that is,

if we take truncates of the Fourier series

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{n=1}^N [a_n \cos(nx) + b_n \sin(nx)]$$

we have  $\lim_{N \rightarrow \infty} \|f - S_N(f)\|_{L^2} = 0$

that is,

$$\lim_{N \rightarrow \infty} \|f - S_N(f)\|_{L^2}^2 = \lim_{N \rightarrow \infty} \int_{-\pi}^{\pi} [f(x) - S_N(f)(x)]^2 dx = 0$$

Question: when does the Fourier series converge for each particular  $x$ ?

- for which  $x$  does  $\lim_{N \rightarrow \infty} S_N(f)(x)$  exist?
- moreover, for which  $x$  does this limit equal  $f(x)$ ?

The Dirichlet kernel is very useful in finding the answers. We will see they help with  $S_N(f)$ .

Def. The Dirichlet kernels are

$$D_N(x) = \sum_{k=-N}^N e^{ikx}$$

or, in terms of trig functions

$$D_N(x) = 1 + \sum_{k=1}^N (e^{ikx} + e^{-ikx}) = 1 + 2 \sum_{k=1}^N \cos(kx)$$

Other normalizations used:

$$\frac{1}{2} \left( 1 + 2 \sum_{k=1}^N \cos(kx) \right)$$

or  $\frac{1}{2\pi} D_N(x) := \delta_N(x)$  (we will work with this)

Simpler formula:

$$D_N(x) = \sum_{k=-N}^N e^{ikx} = e^{-iNx} \sum_{k=0}^{2N} e^{i(k+N)x}$$

(Recall  $1+a+a^2+\dots+a^n = \frac{1-a^{n+1}}{1-a}$ )

$$= e^{-iNx} \frac{1 - e^{i(2N+1)x}}{1 - e^{ix}} = \frac{e^{-iNx} - e^{i(N+1)x}}{1 - e^{ix}}$$

$$= \frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-ix/2} - e^{ix/2}} = \frac{\sin(N+\frac{1}{2})x}{\sin \frac{x}{2}}$$

$$\delta_N(x) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin \frac{x}{2}}$$

## Properties of $\delta_N(x)$ :

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- are  $2\pi$ -periodic (since all  $e^{inx}$  are)
- are even functions (since  $\cos$  is even)
- $\delta_N(x) = 0$  for  $(N + \frac{1}{2})x = k\pi$  ( $k \in \mathbb{Z}$ )

and choosing the zeroes in  $[-\pi, \pi]$  they turn out to be  $\frac{2k\pi}{2N+1}$  for  $k=0, \pm 1, \dots, \pm N$  ( $2N+1$  in all)

- since between any 2 zeroes there must be at least one point of max or min, and it turns out there is exactly one, there are  $N$  max and  $N$  min

Given:  $f \in L^2(-\pi, \pi) \rightarrow \mathbb{R}$

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$$

$$S_N(f)_{(x)} = \sum_{n=-N}^N \hat{f}_n e^{inx} = \frac{a_0}{2} + \sum_{n=1}^N [a_n \cos(nx) + b_n \sin(nx)]$$

↑  
symmetric sum,  
to get a real expression.

Note:  $S_N(f)$  is a trigonometric polynomial, and its Fourier coefficients are  $(S_N f)_n = \begin{cases} \hat{f}_n & \text{if } -N \leq n \leq N \\ 0 & \text{if } |n| > N \end{cases}$

So the sequence  $\{\hat{f}_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$  and it is a product

$$\hat{S}_N f = \hat{f} \cdot \hat{g}_N \quad \text{where } (\hat{g}_N)_n = \begin{cases} 1 & \text{if } -N \leq n \leq N \\ 0 & \text{if } |n| > N \end{cases}$$

in the sense that  $(S_N f)_n = \hat{f}_n \hat{g}_N$

Now:  $\hat{g}_N$  does not depend on  $f$ ! It is a universal cut-off sequence.

Going back to functions, it means that

$S_N(x) =$  "some operation" with  $f$  and this universal

function  $g_N$

$$\text{where } g_N(x) = \sum_{n=-\infty}^{\infty} (\hat{g}_N)_n e^{inx} = \sum_{n=-N}^N e^{inx} = D_N(x) \quad !!!$$

This operation is called convolution.

Indeed.

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$$\int_{-\pi}^{\pi} f(s) D_N(x-s) ds =$$

$$\int_{-\pi}^{\pi} \left( \sum_{k=-\infty}^{\infty} \hat{f}_k e^{iks} \right) \left( \sum_{l=-N}^N e^{il(x-s)} \right) ds$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-N}^N \hat{f}_k e^{ikx} \underbrace{\int_{-\pi}^{\pi} e^{i(k-l)s} ds}_{\substack{=0 \text{ for } k \neq l \\ =2\pi \text{ for } k=l}}$$

$$\begin{aligned} &= 0 \text{ for } k \neq l \\ &= 2\pi \text{ for } k=l \end{aligned}$$

$$= 2\pi \sum_{k=-N}^N \hat{f}_k e^{ikx} = 2\pi S_N(f)(x)$$

$$S_0 S_N(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) D_N(x-s) ds$$

$$= \int_{-\pi}^{\pi} f(s) \cdot \delta_N(x-s) ds$$

$$\Rightarrow S_N(f) = f * \delta_N$$

(the reason to work with  $\delta_N$  rather than  $D_N$ ;  
or, in Stein's book, the convolution is normalized,  
be careful)

[All this trouble is due to the fact that we  
work with  $e^{inx}$  with  $\|e^{inx}\|_{L^2} = \sqrt{2\pi}$ .

If we worked instead with the orthonormal basis

$\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_n$  then we do not need to worry

about normalization and the Fourier transform is  
a unitary operator. But it has been entrenched  
otherwise, and habits (good or bad) are hard to break. 0.40]

Definition: Given 2 functions  $f, g$  on an interval  $[a, b]$   
then their convolution is

$$(f * g)(x) = \int_a^b f(x-s)g(s)ds.$$

### Properties of convolution

Assuming  $f, g$  are nice functions (say, continuous on  $[a, b]$ ) then

$$f * g = g * f \quad \rightarrow \text{commutative}$$

$$f * (g + h) = f * g + f * h$$

$$f * (cg) = c(f * g) \quad (\text{for } c = \text{constant}) \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{bilinear}$$

$$f * (g * h) = (f * g) * h \quad \rightarrow \text{associative}$$

Exercise: prove these

Theorem

And, most importantly, for  $f, g$   $2\pi$ -periodic,

$$\widehat{f * g} = 2\pi \hat{f} \hat{g}$$

(The  $2\pi$  is the price to pay since  $\|e^{inx}\|_{L^2} = \sqrt{2\pi} \neq 1$ )

Proof

$$(\widehat{f * g})_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} (f * g)(x) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx e^{-inx} \int_{-\pi}^{\pi} f(s) g(x-s) ds$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} ds f(s) g(x-s) e^{-in(x-s)} e^{-ins} ds$$

$$x-s=y$$



$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma e^{-i n \sigma} f(\sigma) \underbrace{\int_{x-\pi}^{x+\pi} dy g(y) e^{-i n y}}_{\text{since } g \text{ is } 2\pi\text{-periodic}}$$

$$= \int_{-\pi}^{\pi} dy g(y) e^{-i n y}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma e^{-i n \sigma} f(\sigma) d\sigma \int_{-\pi}^{\pi} dy g(y) e^{-i n y}$$

$$= 2\pi \hat{f}_n \hat{g}_n \quad \square$$

Corollary

$$S_N(f) = f * \delta_N$$

$$\text{or } \sum_{n=-N}^N \hat{f}_n e^{inx} = \int_{-\pi}^{\pi} f(\sigma) \delta_N(x-\sigma) d\sigma$$

Corollary to Corollary

The linear operator  $P: L^2(-\pi, \pi) \rightarrow L^2(-\pi, \pi)$

$$Pf = f * \delta_N$$

is the orthogonal projection onto  $\text{Sp}\{1, e^{\pm ix}, \dots, e^{\pm iNx}\} (= \text{Sp}\{1, \cos x, \sin x, \dots, \cos Nx, \sin Nx\})$

Therefore  $f * \delta_N$  gives the best approximation of  $f$  (in least mean square sense) of  $f$  by trigonometric polynomials in  $\cos kx, \sin kx, k=1, 2, \dots, N$ .