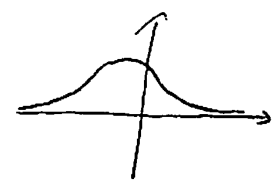
<u>Recall</u>: choose a domain Ω of interest for your problem (e.g. in $n=1$ $\Omega=(a,b)$, or $\Omega=\mathbb{R}$, or...)

A test function on Ω is φ, a C^∞ function on Ω, which vanishes on $\partial\Omega$, and so do all $\varphi^{(k)}$.

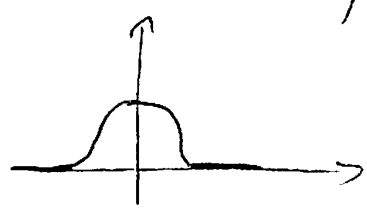
Favourite types of test functions.

$\mathcal{S}$ = rapidly decaying e.g. $\varphi(x)=e^{-x^2}$



$\mathcal{D}$ = with compact support

(φ becomes $\equiv 0$ <u>before</u> $\partial\Omega$) e.g. $\varphi(x)=\begin{cases} e^{-\frac{1}{1-x^2}}, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$



The main point is that one integrates by parts all the boundary terms vanish.

<u>Distribution</u> = linear, continuous functional

$u(\varphi) \equiv (\varphi, u)$

linear $(c\varphi + d\psi, u) = c(\varphi,u) + d(\psi,u)$

continuous : if $\varphi_n \to \varphi$ then $(\varphi_n, u) \to (\varphi, u)$

<u>Function-type distributions</u> If f is integrable then f will be considered also a distribution, defined as

$(\varphi, f) = \int_\Omega \varphi(x) f(x)\,dx$ for any test function φ.

<u>Remark</u> When we work with real-valued functions, then (φ, f) is also the L^2-inner product

But in the complex case we are forced to define distributions as <u>conjugate-linear</u> functionals namely $(c\varphi + d\psi, u) = \bar{c}(\varphi, u) + \bar{d}(\psi, u)$ for all test functions φ, ψ

It is easy to make the change, since if u is linear then u^c defined as $(\varphi, u^c) = (\bar{\varphi}, u)$ is conjugate-linear. We will do this switch when it becomes necessary (so that we can use the full force of Hilbert Spaces theory)

<u>Recall</u> We say that $u_n \to u$ if $(\varphi, u_n) \longrightarrow (\varphi, u)$ for all φ

<u>Recall</u> All distributions are differentiable.

<u>Note</u>: suppose f is differentiable, with f' nice.

Then $(\varphi, f') = \int_a^b \varphi(x) f'(x)dx \underset{\text{by parts}}{=} \varphi(b)f(b) - \varphi(a)f(a) - \int_a^b \varphi'(x)f(x)dx$

$= -\int_a^b \varphi'(x)f(x)\,dx = -(\varphi', f)$ since $\varphi(a) = 0$, $\varphi(b) = 0$

since φ is a test function on $[a, b]$

Define u' for any distribution u by

$$(\varphi, u') := -(\varphi', u) \quad \text{for all test functions } \varphi$$

Examples

- if u is C^1 then u' in the sense of distributions is the same as in the sense of functions.

- Let $u(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ Find u' on $\mathbb{R}$

$$(\varphi, u') = -(\varphi', u) = -\int_{-\infty}^{\infty} \varphi'(x)\,|x|\,dx = \int_{-\infty}^{0} \varphi'(x)\,x\,dx - \int_{0}^{\infty} \varphi'(x)\,x\,dx$$

$$= \underbrace{\varphi(x)\,x\Big|_{-\infty}^{0}}_{=0} - \int_{-\infty}^{0} \varphi(x)\,dx - \underbrace{\varphi(x)\,x\Big|_{0}^{\infty}}_{=0} + \int_{0}^{\infty} \varphi(x)\,dx = \int_{-\infty}^{\infty} \varphi(x)\,A(x)\,dx = (\varphi, A)$$

where $A(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$ so $u' = A$ as it should!

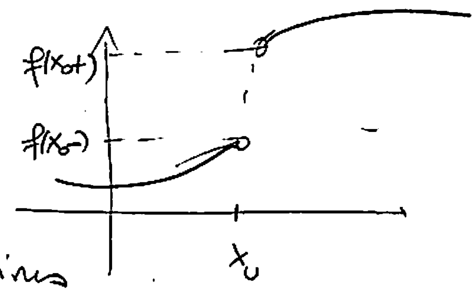
$\left[\text{Note: } A = \frac{1}{2}(1+H)\right]$

- Let $H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$ Find H'

$$(\varphi, H') = -(\varphi', H) = -\int_{-\infty}^{\infty} \varphi'(x)\,H(x)\,dx = -\int_{0}^{\infty} \varphi'(x)\,dx = \varphi(0)$$

$$= (\varphi, \delta) \qquad \underline{\underline{H' = \delta}}$$

More generally, if $f(x)$ is C^1 for all $x \neq x_0$ and $f(x_0+)$, $f(x_0-)$ exist (denote $\tilde{f}(x) = f(x)$ for all $x \neq x_0$) then



f' in the sense of distributions is $f'(x) = \tilde{f}'(x) + \underbrace{[f(x_0+) - f(x_0-)]}_{= \text{mag. of the jump}} \delta(x - x_0)$

Indeed: $(\varphi, f') := -(\varphi', f) = -\int_{-\infty}^{\infty} \varphi'(x) f(x)\,dx$

$$= -\int_{-\infty}^{x_0} \varphi'(x) f(x)\,dx - \int_{x_0}^{\infty} \varphi'(x) f(x)\,dx =$$

$$= -\varphi(x) f(x)\Big|_{-\infty}^{x_0} + \int_{-\infty}^{x_0} \varphi(x) f'(x)\,dx - \varphi(x) f(x)\Big|_{x_0}^{\infty} + \int_{x_0}^{\infty} \varphi(x) f'(x)\,dx$$

$$= -\varphi(x_0) f(x_0-) + \int_{-\infty}^{x_0} \varphi(x) \tilde{f}'(x)\,dx + \varphi(x_0) f(x_0+) + \int_{x_0}^{\infty} \varphi(x) \tilde{f}'(x)\,dx$$

$$= [f(x_0+) - f(x_0-)]\,\varphi(x_0) + \int_{-\infty}^{\infty} \varphi(x) \tilde{f}'(x)\,dx$$

$$= \left(\varphi, [f(x_0+) - f(x_0-)]\,\delta(x - x_0) + \tilde{f}'\right) \text{ for all } \varphi. \quad \square$$

<u>Lemma</u> If $u' = 0$ then $u = \text{const}$

We can multiply distributions by C^∞ (nice) functions:

If $f \in C^\infty$, u distrib, define fu by

$$(\varphi, fu) = (\varphi f, u)$$

(well defined since φf is also a test function)

Convolution If ψ is a test function, we can define $\psi * u$

by $$(\psi * u)(x) = \int \psi(x-t)u(t)\,dt = (\psi(x-\cdot), u(\cdot))$$

for each x consider the test function in t, $\psi(x-t)$ and apply u to it.

Also $\psi * u$ is C^∞! Convolution is regularizing (smoothing)

Example $\boxed{\psi * \delta = \psi}$

since
$$(\psi * \delta)(x) = (\psi(x-\cdot), \delta(\cdot)) = \psi(x))$$

For this reason we write $\psi(x) = \int \psi(x-t)\,\delta(t)\,dt$

Since any distribution is differentiable, then any distribution is infinitely many times differentiable:

$$(\varphi, \frac{du}{dx}) = -(\frac{d\varphi}{dx}, u),$$

$$(\varphi, \frac{d^2u}{dx^2}) = -(\frac{d\varphi}{dx}, \frac{du}{dx}) = (\frac{d^2\varphi}{dx^2}, u)$$

$$\vdots$$

$$(\varphi, \frac{d^ku}{dx^k}) = (-1)^k(\frac{d^k\varphi}{dx^k}, u)$$

Linear differential operators, say :

$$L = p(x)\frac{d}{dx} + q(x) \quad , \; Lu = p(x)\frac{du}{dx} + q(x)u$$

or

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x), \; Lu = -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u$$

Can be defined when u is a distribution as well.

And we can talk about distribution solutions.

Calculus with distributions and with operators in L^2

Illustration of ideas on an example.

We want to study an equation, say $-p(x)u'' - p'(x)u' + q(x)\,u = f(x)$

$$u(0)=0 \qquad p(x)>0$$
$$u(1)=0 \qquad p, p', q \text{ continuous}$$

We could regard it as an operator in $L^2(0,1)$

$$L(x,\tfrac{d}{dx}) = -p(x)\frac{d^2}{dx^2} - p'(x)\frac{d}{dx} + q(x) \equiv -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

on $D(L) = \{ u \in L^2 \mid u', u'' \in L^2,\ u(0)=0,\ u(1)=0 \}$

principle: the problem is modified so that BC become homogeneous, then they are included in the definition of the domain of L.

Then we want to solve: find $u \in D(L)$ so that $Lu = f$.

Or we could formulated in terms of distributions:

find the distributions u so that $Lu=f$ & BC in some sense.

These two formulations can fit together as follows.

I. Real valued functions & equations.

u is a sol. in distrib. of $Lu=f$ means that

$(\varphi, Lu) = (\varphi, f)$ for any test function $\varphi \in C_0^\infty(0,1)$

(recall: this means that $\varphi(x)=0$ outside some $[a,b] \subset (0,1)$)

But by definition, (φ, Lu) means

$$(\varphi, Lu) = \left(\varphi, -\frac{d}{dx}\left(p(x)\frac{d}{dx}u\right) + q(x)u\right) = \left(\varphi, -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)\right) + (\varphi, q(x)u)$$

$$= \left(\frac{d\varphi}{dx}, p(x)\frac{du}{dx}\right) + (q(x)\varphi, u) = \left(p(x)\frac{d\varphi}{dx}, \frac{du}{dx}\right) + (q(x)\varphi, u)$$

$$= \left(-\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right), u\right) + (q(x)\varphi, u)$$

So u is a distribution solution of

$$\left(-\frac{d}{dx}\left(p\frac{d\varphi}{dx}\right) + q\varphi,\ u\right) = (\varphi, f) \quad \text{for all test functions } \varphi \quad (*)$$

A bit of analysis shows that $u(x)$ must be a true function

so if $\int_0^1 \left[-\frac{d}{dx}\left(p\frac{d\varphi}{dx}\right) + q\varphi\right] u(x)\,dx = \int_0^1 \varphi(x) f(x)\,dx$ for all φ

meaning, in terms of $L^2(0,1)$, that

$$\langle L\varphi, u\rangle = \langle \varphi, f\rangle \quad \text{for all } \varphi \in C_0^\infty(0,1)$$

Since $D(L)$ is dense in $L^2(0,1)$ the above equality holds for any $\varphi \in D(L)$

What if we look for Green's functions, namely $f(x) = \delta(x-t)$?
Since $\delta \notin L^2$ then $(*)$ cannot be written in terms of the inner product of L^2.

Recall δ is a limit of functions: there are sequences $\gamma_n \in C_0^\infty$ so that $\gamma_n \to \delta$ in the sense of distributions, namely

$$\int \varphi \gamma_n \to \varphi(0) \text{ for all } \varphi$$

More generally, <u>Theorem</u> any distribution is a limit of C_0^∞ func
(C_0^∞ is dense in the space of distributions)

<u>Then</u> the $\langle\,,\rangle$ of L^2 can be extended, by taking limits, to the duality $(\,,\,)$ in the sense of distributions

This is why it is very justifiable to write $\int f(x)\delta(x)\,dx$
and we mean as a $\lim_{n\to\infty} \int f(x)\gamma_n(x)\,dx$

II For complex-valued functions

When working with differential equations with complex coefficients one needs to work with complex-valued functions and their associated distributions.

The test functions will be defined on intervals in $\mathbb{R}$, but are complex valued. Note that if f is an integrable function, then as a distribution (say, on $\mathbb{R}$)

$$\underbrace{(f,\varphi)}_{\text{duality}} = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx = \underbrace{\langle \bar{f},\varphi\rangle_{L^2(\mathbb{R})}}_{\text{inner product}} \quad \text{for any } \varphi \text{ test function}$$

To cure this discrepancy between $(\,,\,)$ and $\langle\,,\,\rangle$ some authors switch to considering distributions as conjugate-linear, rather than linear functionals.

Here we will keep these two as they are.

<u>Notation</u> If $\mathcal{H} = L^2(I, w(x)dx)$ with $w(x)\geq 0$ (and real) we denote

$$\langle f,g\rangle_w = \int_I \overline{f(x)}\, g(x) w(x)\,dx$$

and note that

$$\langle f,g\rangle_w = (\bar{f}w, g) = (\bar{f}, gw)$$

<u>Dirac's δ</u> Note that all test functions belong to $\mathcal{H} = L_w = L^2(I, w(x)dx)$ and form a dense set.

Let f_n be a fd seq for δ, that is, f_n are test functions (real valued)

and $(f_n,\varphi) \longrightarrow (\delta,\varphi)$ for any φ test.

$$\Rightarrow \langle \bar{f}_n,\varphi\rangle_w = \langle f_n,\varphi\rangle_w \longrightarrow \varphi(0) := \langle \delta,\varphi\rangle$$

So, in the sense of limits, we can speak about $\langle\delta,\varphi\rangle$ and $\delta = \bar{\delta}$.