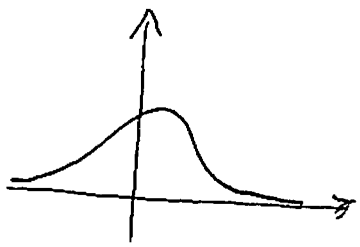
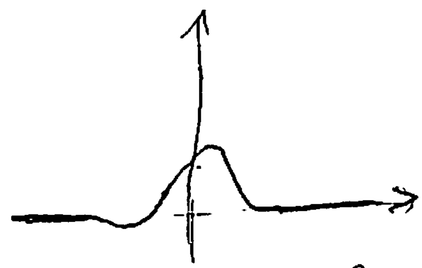
Distributions

We want to talk about Dirac's δ function, which is not a function, but a distribution.

Distributions are linear functionals on a space of test functions — super-nice functions, having all the properties we may wish: infinitely many times diff, quickly decreasing at $+\infty$ and $-\infty$, and so are all their derivatives. E.g. $\mathcal{S}$. Or, another popular choice is $C_0^\infty(\mathbb{R})$ = inf diff functions on $\mathbb{R}$, $\equiv 0$ outside some $[-M,M]$



$\varphi \in \mathcal{S}$



$\varphi \in C_0^\infty(\mathbb{R})$

u is a distribution on a (linear space of) test functions φ if $\cdot\ u(\varphi) \underset{\substack{\uparrow \\ \text{denoted}}}{=:} (u,\varphi) \in \mathbb{R}$ or $\mathbb{C}$ is linear $(u, c_1\varphi_1 + c_2\varphi_2) = c_1(u,\varphi_1) + c_2(u,\varphi_2)$ functional

and continuous: if $\varphi_n \underset{(n\to\infty)}{\longrightarrow} \varphi$ [in a very strong sense, uniformly on any $[-M,M]$ and so do all their derivatives ~ throw in all you wishes]

then $(u,\varphi_n) \longrightarrow (u,\varphi)$ $(n\to\infty)$

Examples

Functions: Take any not too wild function, {say $f(x)=1$, $f(x)=\sin x$, $f(x)=e^{ix}$, $f(x)=H(x)$}, then f defines a distribution by

$$(f,\varphi) := \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$$

Integral converges because φ decays very, very rapidly at $\pm\infty$

The δ-function

Define δ on $\mathcal{S}$ (or $C_0^\infty(\mathbb{R})$) by

$(\delta, \varphi) = \varphi(0)$ linear? Yes | Notation $(\delta,\varphi)=\int_{-\infty}^{\infty}\delta(x)\varphi(x)\,dx = \varphi(0)$
cont? Yes

<u>Shifted δ</u> $(\delta(x-a), \varphi) = \int_{-\infty}^{\infty}\delta(x-a)\varphi(x)\,dx = \int_{-\infty}^{\infty}\delta(y)\varphi(y+a)\,dy = \varphi(a)$

<u>Any distribution is differentiable!</u>

<u>Examples</u>: if my distribution is a nice differentiable function $f(x)$ what should its derivative be?

$$(f'(x), \varphi) = \int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx \underset{\text{by parts}}{=} \underbrace{f(x)\varphi(x)\Big|_{-\infty}^{\infty}}_{\substack{= \\ 0}} - \int_{-\infty}^{\infty} f(x)\,\varphi'(x)\,dx$$

since $f'(x)$ is a function

0 since $\varphi(\pm\infty)=0$

$$= -\int_{-\infty}^{\infty} f(x)\,\varphi'(x)\,dx = -(f, \varphi')$$

since φ' is also a test function.

<u>Definition</u> The derivative of a distribution u is defined as the distribution u' given by

$$(u', \varphi) = -(u, \varphi') \text{ for any test function } \varphi$$

<u>Exercises</u>

1. Show that $|x|$ is differentiable as a distribution and $\frac{d}{dx}|x| = \begin{cases} 1, & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$

2. Show that the derivative of $H(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x<0 \end{cases}$ is δ

3. Show that $(\delta', \varphi) = -\varphi'(0)$

The Fourier transform of a distribution

Examples If f is a super-nice function, say $f \in \mathcal{S}$

then $(\mathcal{F}f, \varphi) = \int_{-\infty}^{\infty} (\mathcal{F}f)(\xi)\varphi(\xi)\,d\xi = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx\,\varphi(\xi)\,d\xi$

$$= \int_{-\infty}^{\infty} f(x) \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\varphi(\xi)\,d\xi}_{\mathcal{F}\varphi}\,dx = (f, \mathcal{F}\varphi)$$

Define for any distribution u on $\mathcal{S}$, $\mathcal{F}u$ by

$$(\mathcal{F}u, \varphi) = (u, \mathcal{F}\varphi)$$

Examples

1. $\mathcal{F}\delta = ?$ Test it: $(\mathcal{F}\delta, \varphi) = (\delta, \mathcal{F}\varphi) = (\mathcal{F}\varphi)(0)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x)\,dx = \left(\frac{1}{\sqrt{2\pi}}, \varphi\right)$$

$\mathcal{F}\delta = \frac{1}{\sqrt{2\pi}}$ (= the constant of normalization in front of $\int_{-\infty}^{\infty} e^{-ix\xi}\varphi(x)\,dx$)

2. $\mathcal{F}\delta(x-a) = ?$

$$(\mathcal{F}\delta(x-a), \varphi) = (\delta(x-a), \mathcal{F}\varphi) = (\mathcal{F}\varphi)(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iax}\varphi(x)\,dx$$

$$= \left(\frac{1}{\sqrt{2\pi}} e^{-iax}, \varphi\right)$$

3. Exercises $\mathcal{F}(e^{ibx}) = ?$

$\mathcal{F}(\sin x) = ?$

$\mathcal{F}(\cos x) = ?$

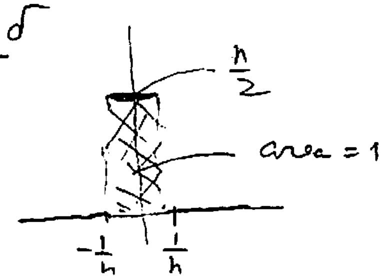
Fundamental sequences for δ

There are many useful sequences of (nice!) functions f_n with the property that $\lim_{n\to\infty} f_n = \delta$

Definition A sequence of distributions $\{u_n\}_n$ converges to the distribution u if $\lim_{n\to\infty} (u_n, \varphi) = (u, \varphi)$ for any test func. φ

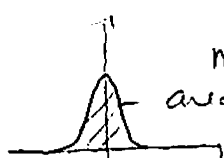
Examples of sequences converging to δ

1. a narrower and narrower step of total area $= 1$



$$\Delta_n = \begin{cases} \frac{n}{2} & \text{if } x \in [-\frac{1}{n}, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

then $\lim_{n\to\infty} \Delta_n = \delta$ check!

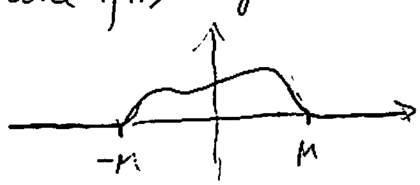
2. 

make it C^∞!

And many others.
Read p. 69, Gerlach

[δ = an impulse concentrated at one point]

Many examples are found by rescaling as follows:
take $f(x)$ a function on $\mathbb{R}$, C^∞, zero outside some $[-M, M]$:
with $\int_{-\infty}^{\infty} f(x)\,dx = 1$



Shrink $[-M, M]$ to $[-M\varepsilon, M\varepsilon]$ while dialating f to keep to total integral $= 1$:

define $f_\varepsilon(x) = \frac{1}{\varepsilon} f(\frac{x}{\varepsilon})$

Then f_ε is zero outside $[-M\varepsilon, M\varepsilon]$ and

$$\int_{-\infty}^{\infty} f_\varepsilon(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\varepsilon} f(\tfrac{x}{\varepsilon})\,dx \underset{y=\frac{x}{\varepsilon}}{\overset{\uparrow}{=}} \int_{-\infty}^{\infty} \frac{1}{\varepsilon} f(y)\,\varepsilon\,dy = 1$$

Then

$\lim_{\varepsilon \to 0} f_\varepsilon = \delta$ since

$$(f_\varepsilon, \varphi) = \int_{-\infty}^{\infty} \frac{1}{\varepsilon} f(\tfrac{x}{\varepsilon}) \varphi(x)\,dx \underset{\frac{x}{\varepsilon}=y}{\overset{\uparrow}{=}} \int_{-\infty}^{\infty} f(y)\varphi(\varepsilon y)\,dy$$

$$= \int_{-M}^{M} f(y)\varphi(\varepsilon y)\,dy = \int_{-M}^{M} \{ f(y)[\varphi(\varepsilon y) - \varphi(0)] + f(y)\varphi(0) \}\,dy$$

$$= \varphi(0) \underbrace{\int_{-M}^{M} f(y)\,dy}_{=1} + \underbrace{\int_{-M}^{M} y f(y) \frac{\varphi(\varepsilon y) - \varphi(0)}{\varepsilon y}}_{\text{mean value thm for the function } \varphi(\varepsilon y)} \varepsilon\,dy$$

$$= \varphi(0) + \underbrace{\varepsilon \int_{-M}^{M} y f(y) \varphi'(\varepsilon c)\,dy}_{\to 0 \text{ as } \varepsilon \to 0} \longrightarrow \varphi(0)$$